Bethe Ansatz

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The problem and the answer

Iterated tensor product — spin chain

$$V^{\otimes N} = V \otimes \ldots \otimes V$$

$V = \mathbb{C}^\ell$

$\{e_1, \ldots, e_\ell\}$

The space $V^{\otimes N}$ has dimension ℓ^N and a natural orthonormal basis

$$e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_N}$$

The symmetric group acts on the tensor product space by permuting coordinates.

For $\sigma \in \mathfrak{S}_N$

$$\mathsf{s}(\sigma)(v_1 \otimes v_2 \ldots \otimes v_N) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(N)}$$

Matrix realization of transpositions

$(E_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}$

$$P_{ij} = \sum_{\alpha,\beta=1}^{\ell} \mathbb{1} \otimes \ldots \otimes E_{\alpha\beta} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes E_{\beta\alpha} \otimes \ldots \otimes \mathbb{1}$$

The action of GL_ℓ on $V^{\otimes N}$

$$g(v_1 \otimes v_2 \ldots \otimes v_N) = gv_1 \otimes gv_2 \ldots \otimes gv_N$$

$$\mathsf{E}_{\alpha\beta} = \sum_{n=1}^{N} E_{\alpha\beta,n}$$

R-matrix

Linear operator on $V \otimes V$

$$R(u) = \frac{u}{u-\eta} \sum_{\alpha,\beta=1}^{\ell} E_{\alpha\alpha} \otimes E_{\beta\beta} - \frac{\eta}{u-\eta} \sum_{\alpha,\beta=1}^{\ell} E_{\alpha\beta} \otimes E_{\beta\alpha}$$

u is a complex (spectral) parameter

Properties

1. Yang-Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v) \qquad \text{on } V \otimes V \otimes V$$

2. Normalization

$$R_{12}(u)R_{21}(-u) = \mathbb{1}$$

3. GL_ℓ-invariance

$$R(u)(g \otimes g) = (g \otimes g)R(u) \qquad \text{for any } g \in \mathrm{GL}_\ell$$

Quantum L-operator and monodromy

$$L(u) = \|L_{\alpha\beta}(u)\|^{\ell}_{\alpha,\beta=1}$$

$$L_1(u) = L(u) \otimes \mathbb{1}$$

$$L_2(v) = \mathbb{1} \otimes L(v)$$

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v)$$

$$L : \quad V \otimes \mathcal{H} \to V \otimes \mathcal{H}$$

$$V \equiv V_{\mathsf{a}} \qquad \mathcal{H} = V_1 \otimes \ldots \otimes V_N$$

auxiliary space *quantum space*

$$L_{\mathsf{a}n} : V_{\mathsf{a}} \otimes V_n \to V_{\mathsf{a}} \otimes V_n$$ *local L − operator*

$$R_{\mathsf{ab}}(u-v)L_{\mathsf{a}n}(u)L_{\mathsf{b}n}(v) = L_{\mathsf{b}n}(v)L_{\mathsf{a}n}(u)R_{\mathsf{ab}}(u-v)$$

Monodromy

$$\mathsf{M}_{\mathsf{a}}(u) = W_{\mathsf{a}} L_{\mathsf{a}1}(u-u_1)\ldots L_{\mathsf{a}N}(u-u_N)$$

complex parameters $u_1, \ldots, u_N$ are called *inhomogeneities*

Transfer matrix of inhomogeneous $\mathfrak{gl}_\ell$ spin chain

$$\mathsf{M}_{\mathsf{a}}(u) = W_{\mathsf{a}} L_{\mathsf{a}1}(u-u_1)\dots L_{\mathsf{a}N}(u-u_N)$$

Commutation relations between entries of monodromy are encoded in

$$R_{\mathsf{ab}}(u-v)\mathsf{M}_{\mathsf{a}}(u)\mathsf{M}_{\mathsf{b}}(v) = \mathsf{M}_{\mathsf{b}}(v)\mathsf{M}_{\mathsf{a}}(u)R_{\mathsf{ab}}(u-v)$$

The transfer matrix $\mathsf{T}(u)$

$$\mathsf{T}(u) = \mathrm{Tr}_{\mathsf{a}}\mathsf{M}_{\mathsf{a}}(u) = \mathrm{Tr}_{\mathsf{a}}\left[W_{\mathsf{a}} L_{\mathsf{a}1}(u-u_1)\dots L_{\mathsf{a}N}(u-u_N)\right]$$

Transfer matrices evaluated at different values of the spectral parameter commute

$$\mathsf{T}(u)\mathsf{T}(v) = \mathsf{T}(v)\mathsf{T}(u)$$

and therefore have a common spectrum

Representations of the L-operator algebra

Let V_n be an irreducible $\mathfrak{gl}_\ell$-module with the highest weight $(m_1^{(n)}, \ldots, m_\ell^{(n)})$

$$L_{\mathsf{a}n}(u) = u\, \mathbb{1}_\mathsf{a} \otimes \mathbb{1}_n - \sum_{\alpha,\beta=1}^{\ell} E_{\alpha\beta} \otimes S^{[\ell]}_{\beta\alpha,n}$$

$S^{[\ell]}_{\alpha\beta,n}$ is a $\mathfrak{gl}_\ell$-generator in the representation $(m_1^{(n)}, \ldots, m_\ell^{(n)})$

$$[S^{[\ell]}_{\alpha\beta,n}, S^{[\ell]}_{\gamma\delta,m}] = \eta\hbar\, \delta_{mn} \big(\delta_{\beta\gamma} S^{[\ell]}_{\alpha\delta,n} - \delta_{\alpha\delta} S^{[\ell]}_{\gamma\beta,n}\big)$$

Spectral problem

Find the common spectrum of commuting transfer matrices
for inhomogeneous $\mathfrak{gl}_\ell$ spin chain

Lieb – Liniger model

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{\hbar^2}{m} c \sum_{i<j}^{N} \delta(q_i - q_j)$$

$c > 0$ the interaction is repulsive

$c < 0$ it is attractive

potential

$$-\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial q_i^2}\psi(q_1,\dots,q_N) + \sum_{i\neq j} v(q_i - q_j)\psi(q_1,\dots,q_N) = E\psi(q_1,\dots,q_N)$$

Due to sufficiently large number of conservation laws scattering is diffractive

The Bethe wave function

$$\psi(q_1,\dots,q_N) = \sum_{\sigma\in\mathfrak{S}_N}\sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{iq_{\sigma(1)}p_{\tau(1)}+\dots+iq_{\sigma(N)}p_{\tau(N)}} \Theta\big(q_{\sigma(1)} < \dots < q_{\sigma(N)}\big)$$

$$\Theta\big(q_{\sigma(1)} < \dots < q_{\sigma(N)}\big) \equiv \prod_{i=1}^{N-1} \Theta\big(q_{\sigma(i+1)} - q_{\sigma(i)}\big)$$

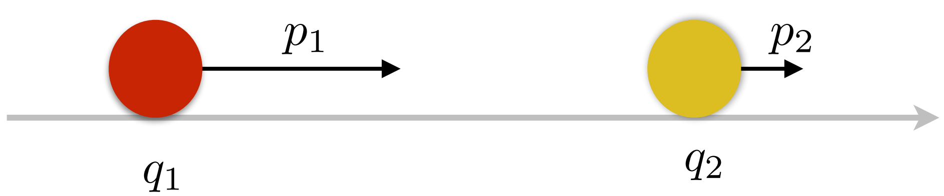
the configuration space $\mathbb{R}^N$ can be divided into $N!$ disconnected domains $\quad q_{\sigma(1)} < q_{\sigma(2)} < \dots < q_{\sigma(N)}$

Relation to physical models

$$\mathcal{A}(\sigma|\sigma_j\tau) = r(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)$$
$$\mathcal{A}(\sigma_j\sigma|\sigma_j\tau) = t(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)$$

Before scattering

$\mathcal{A}(12|12)e^{i(p_1q_1+p_2q_2)}$



After scattering

$\mathcal{A}(12|21)e^{i(p_2q_1+p_1q_2)}$

Reflection

$\mathcal{A}(21|21)e^{i(p_1q_1+p_2q_2)}$

Transmission

Relation to physical models

$$\mathit{For\ Lieb-Liniger\ model} \qquad \mathfrak{r} = -\frac{\eta}{p_1 - p_2 + \eta}, \qquad t = \frac{p_1 - p_2}{p_1 - p_2 + \eta}$$

$$\mathcal{A}(\sigma|\sigma_j\tau) = \mathfrak{r}(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)$$
$$\mathcal{A}(\sigma_j\sigma|\sigma_j\tau) = t(p_{\tau(j)}, p_{\tau(j+1)})\mathcal{A}(\sigma|\tau)$$

$$\longrightarrow \quad \Phi(\tau) \equiv \{\mathcal{A}(\sigma|\tau),\ \sigma \in \mathfrak{S}_N\}$$

Yang's operator

$$Y_j(p_1, p_2) = \mathfrak{r}(p_1, p_2)\,\mathbb{1} + t(p_1, p_2)\,\mathcal{L}(\sigma_j)$$

$$\Phi(\sigma_j\tau) = Y_j(p_{\tau(j)}, p_{\tau(j+1)})\Phi(\tau)$$

$$\Phi(\sigma_j\sigma_{j+1}\sigma_j) = \Phi(\sigma_{j+1}\sigma_j\sigma_{j+1}) \longleftarrow \mathit{Yang-Baxter\ equation}$$

$$S_{ij}(p_1, p_2) = \mathcal{L}(\sigma_{ij})Y_{ij}(p_1, p_2) \qquad Y_j \equiv Y_{jj+1}$$

S – matrix

Relation to physical models

- The two-body S-matrix of the Lieb-Liniger model coincides with the (inverse) R-matrix.
- Particle momenta play the role of inhomogeneities
- Periodicity condition for the Bethe wave function implies that $\Phi(e)$ is a common eigenstate of N matrix operators T_j

$$T_j \, \Phi(e) = \Lambda_j \Phi(e) \,,$$

Transfer matrix!

where

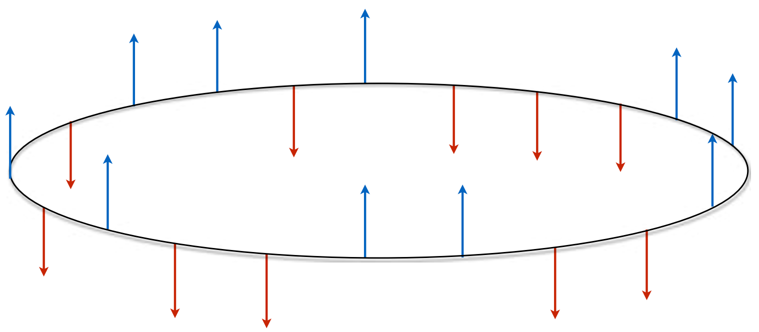
$$T_j = S_{j+1\,j} S_{j+2\,j} \ldots S_{Nj} \cdot S_{1j} \ldots S_{j-1\,j} = \mathsf{T}(p_j) \,,$$

- Once a common eigenvalue, which is a function of momenta, is found, one is left to solve a system of *scalar Bethe equations*

$$\Lambda_j = e^{iLp_j}$$

to determine momenta p_j.

$\mathfrak{gl}_\ell$ magnetic chain



Hamiltonian → $$H = \sum_{n=1}^{N} P_{nn+1}$$

Transfer matrix → $$\mathsf{T}(u) = \mathrm{Tr}_{\mathsf{a}} L_{\mathsf{a}1}(u) \ldots L_{\mathsf{a}N}(u)$$

$$\mathsf{T}(u) = \ell u^N + \sum_{j=0}^{N-1} I_j u^j$$

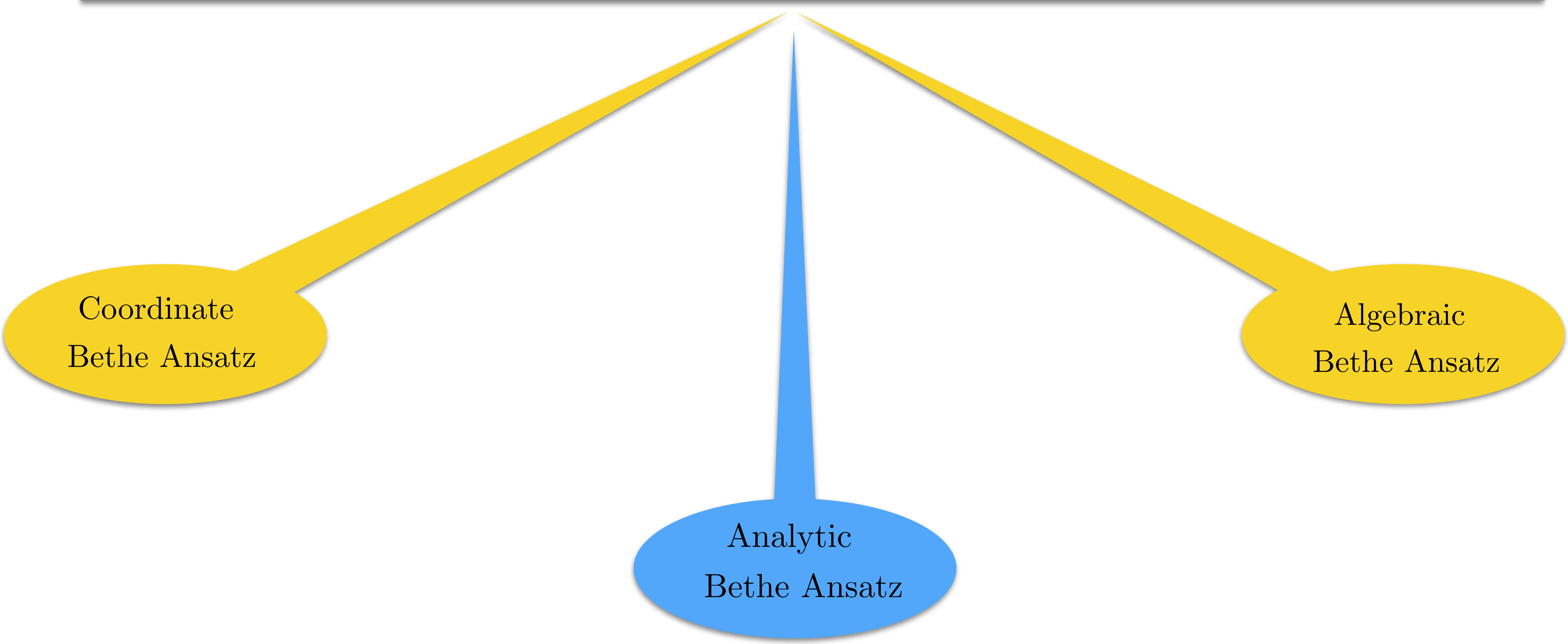
$$H = -\eta\hbar \frac{d\mathsf{T}(u)}{du} T(u)^{-1}\Big|_{u=0} = -\eta\hbar I_1 I_0^{-1}$$

diagonalization of $\mathsf{T}(u)$ leads to diagonalization of H

$\mathfrak{gl}_2$ chain is the XXX Heisenberg model $$H = -J\sum_{n=1}^{N} S_n^\alpha S_{n+1}^\alpha \qquad S_n^\alpha = \tfrac{1}{2}\sigma^\alpha$$

Spectral problem

Find the common spectrum of commuting transfer matrices
for inhomogeneous $\mathfrak{gl}_\ell$ spin chain



How solution looks like

Inhomogeneous $\mathfrak{gl}_\ell$ spin chain with local spin in $(\mathsf{m}_1^{(n)}, \ldots, \mathsf{m}_\ell^{(n)})$ at nth site

$$\mathsf{T}(u)|\Phi\rangle = \Lambda(u)|\Phi\rangle$$

Vacuum polynomials $$\mathsf{p}_k(u) = \prod_{j=1}^{N} \left(u - u_j - \eta \mathsf{m}_k^{(j)}\right), \qquad k = 1, \ldots, \ell\,.$$

Baxter's Q-polynomials $$Q_k(u) = \prod_{j=1}^{\boldsymbol{M}_k} \left(u - u_j^{(k)}\right), \qquad k = 1, \ldots, \ell - 1.$$

,

Bethe equations $$\frac{\mathsf{p}_k(u_j^{(k)})}{\mathsf{p}_{k+1}(u_j^{(k)})} = -\frac{Q_{k-1}(u_j^{(k)})}{Q_{k-1}^{[--]}(u_j^{(k)})}\frac{Q_k^{[--]}(u_j^{(k)})}{Q_k^{[++]}(u_j^{(k)})}\frac{Q_{k+1}^{[++]}(u_j^{(k)})}{Q_{k+1}(u_j^{(k)})} \qquad \begin{aligned} k &= 1, \ldots, \ell - 1 \\ j &= 1, \ldots, \boldsymbol{M}_k \end{aligned}$$

$$Q_k^{[\pm \cdots \pm]}(u) = Q_k(u \pm \tfrac{\eta}{2} s)$$

$$\Lambda(u) = \sum_{k=1}^{\ell} Z_k(u)\,, \qquad Z_k(u) = \mathsf{p}_k(u)\frac{Q_{k-1}^{[--]}(u)}{Q_{k-1}(u)}\frac{Q_k^{[++]}(u)}{Q_k(u)}$$

Eigenvalues of $\mathsf{M}(u)$

Nested Bethe Equations

$$\prod_{l\neq k}^{\boldsymbol{M}_1}\frac{u_k^{(1)}-u_l^{(1)}-\eta}{u_k^{(1)}-u_l^{(1)}+\eta}\prod_{r=1}^{\boldsymbol{M}_2}\frac{u_k^{(1)}-u_r^{(2)}+\eta}{u_k^{(1)}-u_r^{(2)}}=\prod_{n=1}^{N}\frac{u_k^{(1)}-u_n-\eta m_1^{(n)}}{u_k^{(1)}-u_n-\eta m_2^{(n)}},$$

$$\prod_{j=1}^{\boldsymbol{M}_{\alpha-1}}\frac{u_k^{(\alpha)}-u_j^{(\alpha-1)}}{u_k^{(\alpha)}-u_j^{(\alpha-1)}-\eta}\prod_{l\neq k}^{\boldsymbol{M}_\alpha}\frac{u_k^{(\alpha)}-u_l^{(\alpha)}-\eta}{u_k^{(\alpha)}-u_l^{(\alpha)}+\eta}\prod_{r=1}^{\boldsymbol{M}_{\alpha+1}}\frac{u_k^{(\alpha)}-u_r^{(\alpha+1)}+\eta}{u_k^{(\alpha)}-u_r^{(\alpha+1)}}=\prod_{n=1}^{N}\frac{u_k^{(\alpha)}-u_n-\eta m_\alpha^{(n)}}{u_k^{(\alpha)}-u_n-\eta m_{\alpha+1}^{(n)}},$$

$$\prod_{j=1}^{\boldsymbol{M}_{\ell-2}}\frac{u_k^{(\ell-1)}-u_j^{(\ell-2)}}{u_k^{(\ell-1)}-u_j^{(\ell-2)}-\eta}\prod_{l\neq k}^{\boldsymbol{M}_{\ell-1}}\frac{u_k^{(\ell-1)}-u_l^{(\ell-1)}-\eta}{u_k^{(\ell-1)}-u_l^{(\ell-1)}+\eta}=\prod_{n=1}^{N}\frac{u_k^{(\ell)}-u_n-\eta m_{\ell-1}^{(n)}}{u_k^{(\ell)}-u_n-\eta m_\ell^{(n)}},$$

$\alpha=2,\dots,\ell-2$

An eigenstate of the transfer matrix $\mathsf{T}(u)$ *with eigenvalue* $\Lambda(u)$
is the highest weight state of $\mathfrak{gl}_\ell$ *with the highest weight* $[M_1,\dots,M_\ell]$

$$M_1=\sum_{n=1}^{N}m_1^{(n)}-\boldsymbol{M}_1,$$

$$M_2=\sum_{n=1}^{N}m_2^{(n)}+\boldsymbol{M}_1-\boldsymbol{M}_2,$$

$$\cdots$$

$$M_{\ell-1}=\sum_{n=1}^{N}m_{\ell-1}^{(n)}+\boldsymbol{M}_{\ell-2}-\boldsymbol{M}_{\ell-1},$$

$$M_\ell=\sum_{n=1}^{N}m_\ell^{(n)}+\boldsymbol{M}_{\ell-1}.$$

Problem : solutions are difficult to classify, unphysical solutions, singular solutions, completeness

Analytic Bethe Ansatz

I. Functional relations and CBR formula

Yangian

$$L_{\alpha\beta}(u) = \delta_{\alpha\beta} + \sum_{n=1}^{\infty} \frac{L_{\alpha\beta}^{(n)}}{u^n}$$

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v)$$

$Y(\mathfrak{gl}_\ell)$ →

$$[L_{\alpha\beta}^{(r+1)}, L_{\gamma\delta}^{(s)}] - [L_{\alpha\beta}^{(r)}, L_{\gamma\delta}^{(s+1)}] = L_{\gamma\beta}^{(r)} L_{\alpha\delta}^{(s)} - L_{\gamma\beta}^{(s)} L_{\alpha\delta}^{(r)}$$

Evaluation modules

$$L = \sum_{\alpha,\beta=1}^{\ell} E_{\alpha\beta} L_{\alpha\beta}\,, \qquad L_{\alpha\beta} = \delta_{\alpha\beta} - \frac{S_{\beta\alpha,n}^{[\ell]}}{u}$$

$$L_{\alpha\beta}^{(1)} = -S_{\beta\alpha,n}^{[\ell]}$$

$$L_{\alpha\beta}^{(n)} = 0 \text{ for } n > 1$$

Fusion

$R(u) = u - \eta P$

$$R_{ij}(u_i - u_j)\mathsf{M}_i(u_i)\mathsf{M}_j(u_j) = \mathsf{M}_j(u_j)\mathsf{M}_i(u_i)R_{ij}(u_i - u_j)$$

u_i different spectral parameters

$$R(u_1,\dots,u_k) \equiv (R_{12})(R_{13}R_{23})\dots(R_{1k-1}\dots R_{k-2\,k-1})(R_{1k}\dots R_{k-1k})$$

$k(k-1)/2\ R-$ matrices

$$R(u_1,\dots,u_k)\mathsf{M}_1(u_1)\dots\mathsf{M}_k(u_k) = \mathsf{M}_k(u_k)\dots\mathsf{M}_1(u_1)R(u_1,\dots,u_k)$$

Choose $u_k - u_{k+1} = \eta$ then

$$R(u_1,\dots,u_k) = \eta^{\frac{k(k-1)}{2}}\left(\prod_{s=1}^{k} s!\right)P^-_{1\dots k}$$

Here $P^-_{1\dots k}$ is an anti-symmetrizer in $(\mathbb{C}^\ell)^{\otimes k}$

$$P^-_{1\dots k}(e_{\alpha_1}\otimes\dots\otimes e_{\alpha_k}) = \frac{1}{k!}\sum_{\tau\in\mathfrak{S}_k}\mathrm{sgn}\tau\, e_{\alpha_{\tau(1)}}\otimes\dots\otimes e_{\alpha_{\tau(k)}}\,,$$

Fusion

$R(u) = u - \eta P$

$$P^-_{1\ldots k} = \frac{1}{k!} \overrightarrow{\prod_{1\leqslant i<j\leqslant k}} \left(1 - \frac{P_{ij}}{j-i}\right)$$

$P_\tau P^-_{1\ldots k} P_\tau = P^-_{1\ldots k}$

for any $\tau \in \mathfrak{S}_k$

Fusion relations

$$P^-_{1\ldots k} \mathsf{M}_1(u) \ldots \mathsf{M}_k(u-(k-1)\eta) = \mathsf{M}_k(u-(k-1)\eta) \ldots \mathsf{M}_1(u) P^-_{1\ldots k}$$

$$P^-_{1\ldots k} \mathsf{M}_k(u+(k-1)\eta) \ldots \mathsf{M}_1(u) = \mathsf{M}_1(u) \ldots \mathsf{M}_k(u+(k-1)\eta) P^-_{1\ldots k}$$

Multiplying from the left by projector $P^-_{1\ldots k}$

$$\underbrace{\mathsf{M}_1(u) \ldots \mathsf{M}_k(u+(k-1)\eta) P^-_{1\ldots k}} = P^-_{1\ldots k} \underbrace{\mathsf{M}_1(u) \ldots \mathsf{M}_k(u+(k-1)\eta) P^-_{1\ldots k}}$$

Eigenstate of $P^-_{1\ldots k}$

$P^-_{1\ldots k}$ commutes with the action of $\mathfrak{gl}_\ell$ in $(\mathbb{C}^\ell)^{\otimes k}$ and projects this action on an irreducible representation corresponding to the Young diagram $[1^k]$, which is the totally anti-symmetric representation of dimension $\frac{\ell!}{k!(\ell-k)!}$

$$\mathsf{M}_\lambda(u) \equiv \mathsf{M}_1(u) \ldots \mathsf{M}_k(u+(k-1)\eta) P^-_{1\ldots k}\,, \qquad \lambda = [1^k]$$

Quantum minors and quantum determinant

$$\sum_{\tau\in\mathfrak{S}_k} \mathrm{sgn}\tau\, \mathsf{M}_{\alpha_{\tau(1)}\beta_1}(u+(k-1)\eta)\dots \mathsf{M}_{\alpha_{\tau(k)}\beta_k}(u) = \sum_{\tau\in\mathfrak{S}_k} \mathrm{sgn}\tau\, \mathsf{M}_{\alpha_1\beta_{\tau(1)}}(u)\dots \mathsf{M}_{\alpha_k\beta_{\tau(k)}}(u+(k-1)\eta)$$

Quantum minor

$$\begin{aligned}\boldsymbol{\Delta}^{\alpha_1\dots\alpha_k}_{\beta_1\dots\beta_k}(u) &= \sum_{\tau\in\mathfrak{S}_k} \mathrm{sgn}\tau\, \mathsf{M}_{\alpha_1\beta_{\tau(1)}}(u)\dots \mathsf{M}_{\alpha_k\beta_{\tau(k)}}(u+(k-1)\eta)\\ &= \sum_{\tau\in\mathfrak{S}_k} \mathrm{sgn}\tau\, \mathsf{M}_{\alpha_{\tau(1)}\beta_1}(u+(k-1)\eta)\dots \mathsf{M}_{\alpha_{\tau(k)}\beta_k}(u)\end{aligned}$$

Quantum determinant

$$\mathrm{det}_q \mathsf{M}(u) = \sum_{\tau\in\mathfrak{S}_\ell} \mathrm{sgn}\tau\, \mathsf{M}_{\alpha_1\beta_{\tau(1)}}(u)\dots \mathsf{M}_{\alpha_\ell\beta_{\tau(\ell)}}(u+(\ell-1)\eta)$$

$$\mathsf{M}_1(u)\dots\mathsf{M}_\ell(u+(\ell-1)\eta)P^-_{1\dots\ell} = \mathrm{det}_q\mathsf{M}(u)\,P^-_{1\dots\ell}$$

Quantum determinant generates center of $\mathrm{Y}(\mathfrak{gl}_\ell)$

$$[\mathsf{M}_{\alpha\beta}(u), \mathrm{det}_q\mathsf{M}(v)] = 0\,, \quad \forall\alpha,\beta = 1,\dots,\ell\,.$$

Coefficients of $\mathrm{qdet}\mathsf{M}(u)$ in the expansion of u are central elements of $\mathrm{Y}(\mathfrak{gl}_\ell)$

Bethe suablagebra of Yangian

$$\mathscr{R}_{\bar{1}\ldots\bar{m};1\ldots k}(u,v)\left[P^{-}_{\bar{1}\ldots\bar{m}}\mathsf{M}_{[\bar{m}]}(u)\right]\left[P^{-}_{1\ldots k}\mathsf{M}_{[k]}(v)\right] = \left[P^{-}_{1\ldots k}\mathsf{M}_{[k]}(v)\right]\left[P^{-}_{\bar{1}\ldots\bar{m}}\mathsf{M}_{[\bar{m}]}(u)\right]\mathscr{R}_{\bar{1}\ldots\bar{m};1\ldots k}(u,v)$$

$$\mathscr{R}_{\bar{1}\ldots\bar{m};1\ldots k}(u,v) = 1 + \sum_{r=1}^{m} \frac{(-1)^r r!\eta^r}{(u-v-(k-1)\eta)\ldots(u-v-(k-r)\eta)} \times \sum_{1\leqslant n_1<\ldots<n_r\leqslant m}\ \sum_{1\leqslant s_1<\ldots<s_r\leqslant k} P_{\bar{n}_1 s_1}\ldots P_{\bar{n}_r s_r}$$

Transfer matrices associated to auxiliary spaces $[1^k]$ of $\mathfrak{gl}_\ell$

$$\mathsf{T}_k(u) = \mathrm{Tr}_{1\ldots k}\left[P^{-}_{1\ldots k}\mathsf{M}_1(u)\ldots\mathsf{M}_k(u+(k-1)\eta)\right] = \mathrm{Tr}_\lambda \mathsf{M}_\lambda(u)$$

$$\lambda = [1^k]$$

$$\mathsf{T}_k(u)\mathsf{T}_m(v) = \mathsf{T}_m(v)\mathsf{T}_k(u)\,, \quad \forall k,m\leqslant \ell$$

Bethe subablebra

C is a constant matrix with simple spectrum

$$\mathsf{T}_k(u,C) = \mathrm{Tr}_{1\ldots k}\left[P^{-}_{1\ldots k}C_1\ldots C_k\mathsf{M}_1(u)\ldots\mathsf{M}_k(u+(k-1)\eta)\right]$$

twist

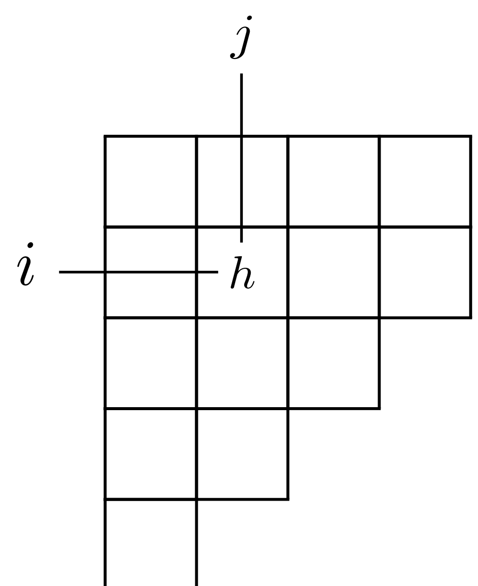
Coefficients of $\mathsf{T}_1(u,C),\ldots,\mathsf{T}_\ell(u,C)$ in the large u expansion are algebraically independent and generate maximal commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_\ell)$

Transfer matrix for an arbitrary $\mathfrak{gl}_\ell$ module

$$T_\lambda(u) = \mathrm{Tr}_{1\ldots k}[P_{t_\lambda}\mathsf{M}_1(u+\eta c_1)\mathsf{M}_2(u+\eta c_2)\ldots\mathsf{M}_k(u+\eta c_k)]$$

P_λ is a projector corresponding to any standard tableau t_λ of shape λ

$c(t_\lambda) = (c_1, c_2, \ldots, c_k)$ is a content vector of t_λ



$\lambda \vdash k$

Content of h

$c(h) = j - i$

Transfer matrices in anti-symmetric representations $T_k(u) \equiv T_{[1^k]}(u)$

There is a unique standard tableau with k boxes and content vector $c = (0, -1, -2, \ldots, 1-k)$

$$T_k(u) = \mathrm{Tr}_{1\ldots k}[P^-_{1\ldots k}\mathsf{M}_1(u)\mathsf{M}_2(u-\eta)\ldots\mathsf{M}_k(u-(k-1)\eta)]$$

Transfer matrices as generalization of characters

$\mathsf{M}(u)$ *monodromy*	$\longleftrightarrow$	$g \in \mathrm{GL}_\ell$ *group element*
$\mathsf{M}_\lambda(u)$ *monodromy in representation* λ	$\longleftrightarrow$	$\rho_\lambda(g)$
$T_\lambda(u)$ *transfer matrix*	$\longleftrightarrow$	$\chi_\lambda(g) = \mathrm{Tr}\,\rho_\lambda(g)$ *group character*
Functional relations among commuting transfer matrices	$\longleftrightarrow$	*Jacobi-Trudi formulae*

Jacobi-Trudy formulae

An irrep of GL_ℓ indexed by partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ has the character

$$\chi_\lambda(g) = s_\lambda(z_1, \ldots, z_\ell)$$

Schur polynomial

$$s_\lambda(z_1, \ldots, z_\ell) = \frac{\det(z_j^{\lambda_i + \ell - i})_{1 \leqslant i,j \leqslant \ell}}{\det(z_j^{\ell - i})_{1 \leqslant i,j \leqslant \ell}}$$

Jacobi's bialternant formula

for GL_3 and symmetric representations

$$\begin{aligned}
s_{\square}(z_1, z_2, z_3) &= z_1 + z_2 + z_3 \,, \\
s_{\square\square}(z_1, z_2, z_3) &= z_1^2 + z_2^2 + z_3^2 + z_1 z_2 + z_1 z_3 + z_2 z_3 \,, \\
s_{\square\square\square}(z_1, z_2, z_3) &= z_1^3 + z_2^3 + z_3^3 + z_1 z_2^2 + z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_3 + z_1 z_3^2 + z_2 z_3^2 + z_1 z_2 z_3 \,,
\end{aligned}$$

and for anti-symmetric representations

$$\begin{aligned}
s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(z_1, z_2, z_3) &= z_1 z_2 + z_1 z_3 + z_2 z_3 \,, \\
s_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(z_1, z_2, z_3) &= z_1 z_2 z_3 \,.
\end{aligned}$$

Another example

$$s_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(z_1, z_2, z_3) = z_1^2 z_2 + z_1^2 z_3 + z_2^2 z_1 + z_2^2 z_3 + z_3^2 z_1 + z_3^2 z_2 + 2 z_1 z_2 z_3$$

Jacobi-Trudy formulae

Combinatorial definition of Schur polynomials

$$s_\lambda(z_1,\dots,z_\ell) = \sum_{t\in\{\mathsf{sst}_\lambda\}} z_1^{n_1(t)}\dots z_\ell^{n_\ell(t)}$$

where the sum runs over a set $\{\mathsf{sst}_\lambda\}$ of all semi-standard Young tableaux of shape λ

$n_i(t)$ is the number of times the symbol i occurs in t.

Example

$$s_{(2,1)}(z_1,z_2,z_3) \quad = z_1^2z_2 + z_1^2z_3 + z_2^2z_1 + z_2^2z_3 + z_3^2z_1 + z_3^2z_2 + 2z_1z_2z_3$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

Tableau sum formula

$$s_\lambda(z_1,\dots,z_\ell) = \sum_{t\in\{\mathsf{sst}_\lambda\}} \prod_{(i,j)\in t} z_{\#(i,j)}$$

$\#(i,j)$ is a number that occurs in this tableau in the box with coordinates (i,j)

Jacobi-Trudy formulae

Tableau sum formula

$$s_\lambda(z_1,\ldots,z_\ell) = \sum_{t\in\{\mathsf{asst}_\lambda\}} \prod_{(i,j)\in t} z_{\#(i,j)}$$

anti-semi-standard Young tableaux of shape λ

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$\{s_\lambda : \lambda \vdash k, l(\lambda) \leqslant \ell\}$ is a linear basis in the space of symmetric polynomials of degree k in ℓ indeterminates

Fundamental bases

$$\det(g-z) = \sum_{k=0}^{\ell} (-z)^{\ell-k} e_k\,, \quad z\in\mathbb{C}\,.$$

Here e_k are *elementary symmetric functions* of the variables z_i

$$e_k = \sum_{1\leqslant i_1<\ldots<i_k\leqslant\ell} z_{i_1}\cdots z_{i_k} \qquad e_0 = 1$$

e_k are characters of elementary anti-symmetric representations $[1^k]$ of GL_ℓ $\qquad e_k = s_{[1^k]}$

Jacobi-Trudy formulae

Complete homogeneous symmetric polynomials

$$\det(g-z)^{-1} = \sum_{k=0}^{\infty} (-1)^{\ell} z^{-\ell-k} h_k \,, \qquad h_0 = 1\,,$$

$$h_k = \sum_{i_1 \leqslant \ldots \leqslant i_k} z_{i_1} \cdots z_{i_k}\,. \qquad\qquad h_k = s_{[k]}$$

The characters e_k and h_k are of special significance, because any Schur polynomial can be expressed in terms of either h_k or e_k by means of *Jacobi-Trudi formulae*

$$s_\lambda = \det\big(h_{\lambda_j+i-j}\big)_{1\leqslant i,j\leqslant \ell}$$

$$s_\lambda = \det\big(e_{\lambda'_j+i-j}\big)_{1\leqslant i,j\leqslant \lambda_1}$$

λ' is conjugate to λ

Cherednik-Bazhanov-Reshetikhin (CBR) formula

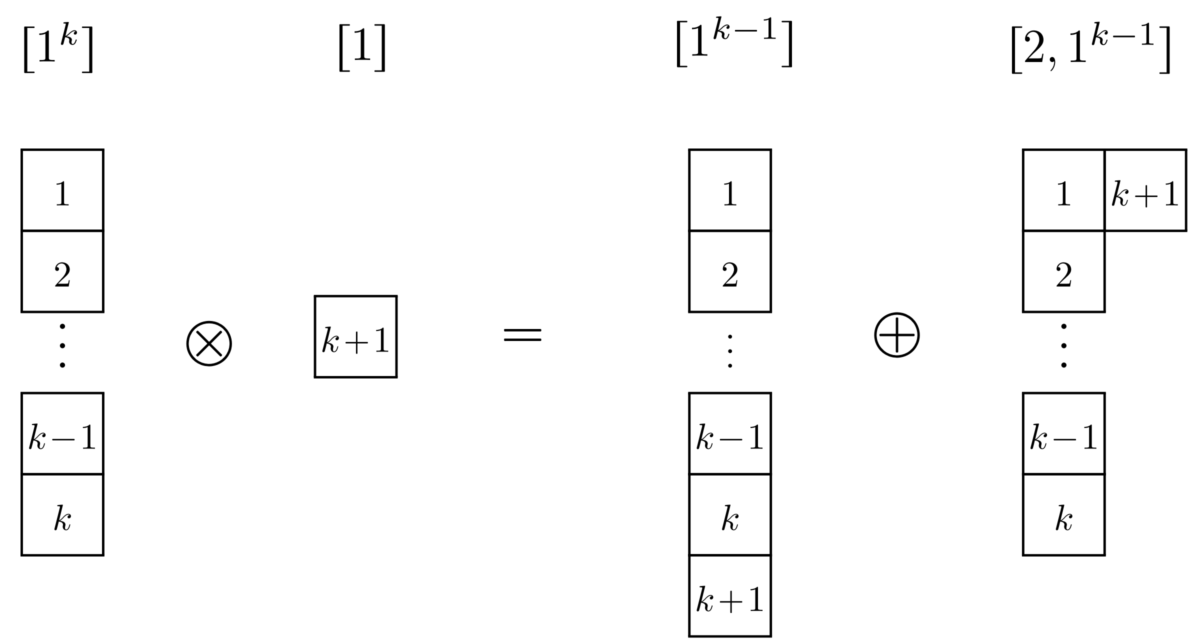
$$s_\lambda = \det\left(e_{\lambda'_j+i-j}\right)_{1\leqslant i,j\leqslant \lambda_1}$$

$$T_\lambda(u) = \det\left[T_{\lambda'_j+i-j}(u+(i-1)\eta)\right]_{1\leqslant i,j\leqslant \lambda_1}$$

The CBR formula can be regarded as a solution of the functional relations among transfer matrices

Functional relations

$$T_k(u)\,T_1(u+\eta) = T_{k+1}(u+\eta) + T_{[2,1^{k-1}]}(u)$$



$$P^-_{1\dots k}\otimes \mathbb{1} = P^-_{1\dots k+1} + P_{\text{hook}}$$

non-primitive idempotent

primitive idempotents

$$T_{[2,1^{k-1}]}(u) = \begin{vmatrix} T_k(u) & T_0(u) \\ T_{k+1}(u+\eta) & T_1(u+\eta) \end{vmatrix} \longrightarrow T_\lambda(u) = \begin{vmatrix} T_{\lambda'_1}(u) & T_{\lambda'_2-1}(u) \\ T_{\lambda'_1+1}(u+\eta) & T_{\lambda'_2}(u+\eta) \end{vmatrix}$$

$\lambda'_1 = k$ and $\lambda'_2 = 1$

CBR formula

Analytic Bethe Ansatz

II. Spectrum of Bethe subalgebra and quantum-classical duality

Analytic structure of transfer matrices

Kinematic zeroes arise due to degeneration of the R-matrix at special values of the spectral parameter

$\mathsf{T}_k(u)$ has $k-1$ kinematic zeroes

$$u = u_j, u_j+\eta, \ldots, u_j+(k-2)\eta$$

We also need zeroes of $\mathsf{T}_{[2^k]}(u) = \mathsf{T}_{k,2}(u)$

$$\mathsf{T}_{k,2}(u) = \mathrm{Tr}\big[P_{[2^k]}\mathsf{M}_1(u)\mathsf{M}_2(u-\eta)\ldots\mathsf{M}_k(u-(k-1)\eta) \\ \times\mathsf{M}_{k+1}(u+\eta)\mathsf{M}_{k+2}(u)\mathsf{M}_{k+3}(u-\eta)\ldots\mathsf{M}_{2k}(u-(k-2)\eta)\big]$$

$$\boxed{u_j\,, u_j+\eta\,, \ldots, u_j+(k-3)\eta}\,, u_j+(k-2)\eta\,,$$

$$u_j-\eta\,, \boxed{u_j\,, u_j+\eta\,, \ldots, u_j+(k-3)\eta}\,,$$

double zeroes

There is one more (less obvious) zero at $u_j+(k-1)\eta$

Talalaev's formula

$$\begin{aligned} \mathsf{T}_k(u) &= T_k(u+(k-1)\eta) \\ &= \mathrm{Tr}_{1\ldots k}\big[P^-_{1\ldots k}\mathsf{M}_1(u+(k-1)\eta)\ldots\mathsf{M}_k(u)\big] \\ &= \mathrm{Tr}_{1\ldots k}\big[P^-_{1\ldots k}\mathsf{M}_1(u)\ldots\mathsf{M}_k(u+(k-1)\eta)\big] \end{aligned}$$

$k-1$ kinematic zeros of $\mathsf{T}_k(u)$ are at

$$u = u_j - \eta, \ldots, u_j - (k-1)\eta$$

Introduce the sift operator $e^{\eta\partial_u}$: $\qquad e^{\eta\partial_u} f(u) = f(u+\eta)$

$$\mathsf{T}_k(u)e^{k\eta\partial_u} = \mathrm{Tr}_{1\ldots k}\big[P^-_{1\ldots k}\mathsf{M}_1(u)e^{\eta\partial_u}\mathsf{M}_2(u)e^{\eta\partial_u}\ldots\mathsf{M}_k(u)e^{\eta\partial_u}\big]$$

$$\begin{aligned} \det\big(\mathsf{M}(u)e^{\eta\partial_u} - z\big) &\equiv \mathrm{Tr}_{1\ldots\ell}P^-_{1\ldots\ell}\big(\mathsf{M}_1(u)e^{\eta\partial_u} - z\big)\big(\mathsf{M}_2(u)e^{\eta\partial_u} - z\big)\ldots\big(\mathsf{M}_\ell(u)e^{\eta\partial_u} - z\big) \\ &= \sum_{k=0}^{\ell} C^k_\ell(-z)^{\ell-k}\mathrm{Tr}_{1\ldots\ell}\, P^-_{1\ldots\ell}\mathsf{M}_1(u)e^{\eta\partial_u}\ldots\mathsf{M}_k(u)e^{\eta\partial_u} \\ &= \sum_{k=0}^{\ell}(-z)^{\ell-k}\mathrm{Tr}_{1\ldots k}\, P_{1\ldots k}\mathsf{M}_1(u)e^{\eta\partial_u}\ldots\mathsf{M}_k(u)e^{\eta\partial_u} \end{aligned}$$

$$\mathrm{Tr}_{k+1\ldots\ell}P^-_{1\ldots\ell} = \frac{k!(\ell-k)!}{\ell!}P^-_{1\ldots k}$$

Talalaev's formula

$$\det\big(\mathsf{M}(u)e^{\eta\partial_u} - z\big) = \sum_{k=0}^{\ell} (-z)^{\ell-k}\mathsf{T}_k(u)e^{k\eta\partial_u}$$

$$\det\big(1 - \mathsf{M}(u)e^{\eta\partial_u}\big) = \sum_{k=0}^{\ell} (-1)^{k}\mathsf{T}_k(u)e^{k\eta\partial_u}$$

$\mathsf{T}_k(u)$ are *quantum characters* corresponding to elementary anti-symmetric representations of GL_ℓ

Lagrange interpolation

Analytic data for $\mathsf{T}_k(u)$

1. $\mathsf{T}_k(u)$ is a polynomial of degree kN

$$\mathsf{T}_k(u) \sim u^{kN}\,\mathrm{Tr}_{1\ldots k} P^-_{1\ldots k} = C^k_\ell u^{kN} \qquad \text{at } u \to \infty$$

2. $\mathsf{T}_k(u)$ has $N(k-1)$ kinematic zeroes at $u_j - n\eta$, $n = 1, \ldots, k-1$ and $j = 1, \ldots, N$

These data are exactly enough to uniquely reconstruct polynomial $\mathsf{T}_k(u)$ provided the values $\mathsf{T}_k(u_j)$ for all $j = 1, \ldots, N$ are given

$$g_k(u) = \prod_{j=1}^{N} \prod_{m=1}^{k-1} (u - u_j + m\eta) \qquad g_1(u) = 1$$

$$\mathsf{T}_k(u) = g_k(u) \left[C^k_\ell \prod_{j=1}^{N} (u - u_j) + \sum_{j=1}^{N} \frac{\mathsf{T}_k(u_j)}{g_k(u_j)} \prod_{s \neq j}^{N} \frac{u - u_s}{u_j - u_s} \right]$$

Lagrange interpolation

$$\mathsf{T}_1(u) = \ell \prod_{j=1}^{N} (u - u_j) + \sum_{j=1}^{N} \mathsf{T}_1(u_j) \prod_{s \neq j}^{N} \frac{u - u_s}{u_j - u_s}$$

Consider the large u expansion

$$\frac{\mathsf{T}_1(u)}{u^N} \sim \ell - \frac{\ell}{u} \sum_{j=1}^{N} u_j + \frac{1}{u} \sum_{j=1}^{N} \mathsf{T}_1(u_j) \prod_{k \neq j}^{N} \frac{1}{u_j - u_k} + \dots$$

Expand $\mathsf{T}_1(u)$ as the trace of monodromy

$$\frac{\mathsf{T}_1(u)}{u^N} \sim \ell - \frac{\ell}{u} \sum_{j=1}^{N} u_j - \frac{\eta}{u} \sum_{j=1}^{N} \mathrm{Tr}_{\mathsf{a}} P_{\mathsf{a}j} + \frac{1}{u^2} \sum_{i<j}^{N} (u_i + \eta)(u_j + \eta) \mathrm{Tr}_{\mathsf{a}} (P_{\mathsf{a}i} P_{\mathsf{a}j}) + \dots$$

$$\sum_{j=1}^{N} \mathrm{Tr}_{\mathsf{a}} P_{\mathsf{a}j} = \sum_{j=1}^{N} \sum_{\alpha=1}^{\ell} E_{\alpha\alpha,n} = N$$

$$\sum_{j=1}^{N} \mathsf{T}_1(u_j) \prod_{k \neq j}^{N} \frac{1}{u_j - u_k} = -N\eta$$

Lagrange interpolation

Quantum determinant

$$\det_q \mathsf{M}(u) = \mathsf{T}_\ell(u) = \mathrm{Tr}_{1\ldots\ell}\big[P^-_{1\ldots\ell}\mathsf{M}_1(u+(\ell-1)\eta)\ldots\mathsf{M}_\ell(u)\big]$$

$$\det_q \mathsf{M}(u) = g_\ell(u)\prod_{j=1}^{N}(u-u_j-\eta) = \prod_{j=1}^{N}(u-u_j-\eta)\prod_{m=1}^{\ell-1}\prod_{j=1}^{N}(u-u_j+m\eta)$$

Fusion relations for $\mathsf{T}_k(u_j)$

$$T_\lambda(u) = \det\left[\mathsf{T}_{\lambda'_j+i-j}(u+(j-\lambda'_j)\eta)\right]_{1\leqslant i,j\leqslant\lambda_1}$$

$$T_{k,2}(u+(k-1)\eta) = \mathsf{T}_k(u)\mathsf{T}_k(u+\eta) - \mathsf{T}_{k-1}(u+\eta)\mathsf{T}_{k+1}(u)$$

zeros $$u_j-\eta, \ \ldots \ , u_j-k\eta\,, \quad \forall j=1,\ldots,N$$

where zeroes at the first and the last location are single, while the remaining zeroes are double

$T_{k,2}(u)$ has yet another zero at $u = u_j + (k-1)\eta$

At $u = u_j$ for any $k \geqslant 1$

$$\mathsf{T}_k(u_j)\mathsf{T}_k(u_j+\eta) - \mathsf{T}_{k-1}(u_j+\eta)\mathsf{T}_{k+1}(u_j) = 0$$

Fusion relations for $\mathsf{T}_k(u_j)$

$$\mathsf{T}_k(u_j)\mathsf{T}_k(u_j+\eta) - \mathsf{T}_{k-1}(u_j+\eta)\mathsf{T}_{k+1}(u_j) = 0$$

$$\begin{aligned}
\mathsf{T}_k(u_j+\eta) &= \frac{\mathsf{T}_{k+1}(u_j)}{\mathsf{T}_k(u_j)}\mathsf{T}_{k-1}(u_j+\eta) = \frac{\mathsf{T}_{k+1}(u_j)}{\mathsf{T}_k(u_j)}\frac{\mathsf{T}_k(u_j)}{\mathsf{T}_{k-1}(u_j)}\mathsf{T}_{k-2}(u_j+\eta) \\
&= \frac{\mathsf{T}_{k+1}(u_j)}{\mathsf{T}_k(u_j)}\frac{\mathsf{T}_k(u_j)}{\mathsf{T}_{k-1}(u_j)}\cdots\frac{\mathsf{T}_2(u_j)}{\mathsf{T}_1(u_j)}\mathsf{T}_0(u_j+\eta) = \frac{\mathsf{T}_{k+1}(u_j)}{\mathsf{T}_1(u_j)}\,.
\end{aligned}$$

$$\boxed{\mathsf{T}_1(u_j)\mathsf{T}_k(u_j+\eta) = \mathsf{T}_{k+1}(u_j)}$$

$k = 1,\ldots,\ell-1$ as $\mathsf{T}_\ell(u_j+\eta) = 0$

Consistency condition of the CBR formula with the analytic structure of fused transfer matrices

Spectral equations

All the quantities $\mathsf{T}_k(u_j)$ pair-wise commute,

and can be viewed as generators of the Bethe subalgebra.

Fusion relations $\mathsf{T}_1(u_j)\mathsf{T}_k(u_j+\eta) = \mathsf{T}_{k+1}(u_j)$ together with Lagrange interpolation formulas for $\mathsf{T}_k(u)$ may be used to express these quantities via $\mathsf{T}_1(u_j)$ in an algebraic way

$$\mathsf{T}_k(u+\eta) = g_k(u+\eta)\left[C_\ell^k \prod_{j=1}^{N}(u-u_j+\eta) + \sum_{l=1}^{N} \frac{\mathsf{T}_k(u_l)}{g_k(u_l)} \prod_{m\neq l}^{N} \frac{u-u_m+\eta}{u_l-u_m}\right]$$

$$g_k(u+\eta) = g_{k+1}(u)\prod_{j=1}^{N}\frac{1}{u-u_j+\eta}$$

$$\mathsf{T}_k(u+\eta) = g_{k+1}(u)\left[C_\ell^k + \sum_{l=1}^{N}\frac{\mathsf{T}_k(u_l)}{g_k(u_l)(u-u_l+\eta)}\prod_{m\neq l}^{N}\frac{1}{u_l-u_m}\right]$$

Introducing an auxiliary function

$$h(u,u_l) = \frac{1}{u-u_l+\eta}\prod_{m\neq l}^{N}\frac{1}{u_l-u_m}$$

Spectral equations

$$\mathsf{T}_k(u+\eta) = g_{k+1}(u)\left[C_\ell^k + \sum_{l=1}^{N} \frac{\mathsf{T}_k(u_l)}{g_k(u_l)} h(u,u_l)\right]$$

$$\downarrow$$

$$\mathsf{T}_k(u_j+\eta) = g_{k+1}(u_j)\left[C_\ell^k + \sum_{l=1}^{N} \frac{\mathsf{T}_k(u_l)}{g_k(u_l)} h(u_j,u_l)\right]$$

$$\mathsf{T}_1(u_j)\mathsf{T}_k(u_j+\eta) = \mathsf{T}_{k+1}(u_j)$$

$$\frac{\mathsf{T}_{k+1}(u_j)}{g_{k+1}(u_j)} = \mathsf{T}_1(u_j)\left[C_\ell^k + \sum_{l=1}^{N} h(u_j,u_l)\frac{\mathsf{T}_k(u_l)}{g_k(u_l)}\right]$$

Recurrence!

$$x_j \equiv \mathsf{T}_1(u_j)$$

$$\frac{\mathsf{T}_2(u_j)}{g_2(u_j)} = x_j\left[C_\ell^1 + \sum_{l=1}^{N} h(u_j,u_l)x_l\right] \qquad g_1(u_j) = 1$$

$$\frac{\mathsf{T}_3(u_j)}{g_3(u_j)} = x_j\left[C_\ell^2 + C_\ell^1\sum_{l=1}^{N} h(u_j,u_l)x_l + \sum_{l=1}^{N}\sum_{n=1}^{N} h(u_j,u_l)h(u_l,u_n)x_l x_n\right]$$

$$\cdots$$

Spectral equations

$$h(u,u_l) = \frac{1}{u-u_l+\eta}\prod_{m\neq l}^{N}\frac{1}{u_l-u_m}$$

$x_j \equiv \mathsf{T}_1(u_j)$

$$\frac{\mathsf{T}_\ell(u_j)}{g_\ell(u_j)} = x_j\Bigg[C_\ell^{\ell-1} + C_\ell^{\ell-2}\sum_{m=1}^{N} h(u_j,u_m)x_m + C_\ell^{\ell-3}\sum_{m_1,m_2=1}^{N} h(u_j,u_{m_1})h(u_{m_1},u_{m_2})x_{m_1}x_{m_2}$$
$$+ \ \ldots + C_\ell^1 \sum_{m_1,\ldots,m_{\ell-2}=1}^{N} h(u_j,u_{m_1})\ldots h(u_{m_{\ell-3}},u_{m_{\ell-2}})x_{m_1}\ldots x_{m_{\ell-2}}$$
$$+ \sum_{m_1,\ldots,m_{\ell-1}=1}^{N} h(u_j,u_{m_1})\ldots h(u_{m_{\ell-1}},u_{m_{\ell-1}})x_{m_1}\ldots x_{m_{\ell-1}}\Bigg].$$

1st vacuum polynomial $\mathsf{p}_1(u) = \prod_{k=1}^{N}(u-u_k-\eta)$

$$\mathsf{p}_1(u_j) = x_j\Bigg[C_\ell^{\ell-1} + C_\ell^{\ell-2}\sum_{m=1}^{N} h(u_j,u_m)x_m + C_\ell^{\ell-3}\sum_{m_1,m_2=1}^{N} h(u_j,u_{m_1})h(u_{m_1},u_{m_2})x_{m_1}x_{m_2}$$
$$+ \ \ldots + C_\ell^1 \sum_{m_1,\ldots,m_{\ell-2}=1}^{N} h(u_j,u_{m_1})\ldots h(u_{m_{\ell-3}},u_{m_{\ell-2}})x_{m_1}\ldots x_{m_{\ell-2}}$$
$$+ \sum_{m_1,\ldots,m_{\ell-1}=1}^{N} h(u_j,u_{m_1})\ldots h(u_{m_{\ell-1}},u_{m_{\ell-1}})x_{m_1}\ldots x_{m_{\ell-1}}\Bigg].$$

N polynomial equations of ℓth order to determine x_j

By the elimination method, one may obtain for any x_j a polynomial equation of order ℓ^N, the latter number equals to dimension of the Hilbert space of the $\mathfrak{gl}_\ell$ spin chain. The roots of this equation are ℓ^N eigenvalues of operator $\mathsf{T}_1(u_j)$

Spectral equations for $\mathfrak{gl}_2$ spin chain

$N \times N$ matrix L

$$L_{ij} = -h(u_i, u_j)x_j = -\frac{x_j}{u_i - u_j + \eta} \prod_{k \neq j}^{N} \frac{1}{u_j - u_k}$$

Spectral equations

$$\frac{\prod_{k=1}^{N}(u_j - u_k - \eta)}{x_j} = \sum_{k=1}^{N} \sum_{m=0}^{\ell-1} (-1)^m C_\ell^{\ell-1-m} (L^m)_{jk}\,, \qquad j = 1, \ldots, \ell\,.$$

For $\mathfrak{gl}_2$

$$\prod_{k=1}^{N}(u_j - u_k - \eta) = 2x_j + x_j \sum_{k=1}^{N} \frac{x_k}{u_j - u_k + \eta} \prod_{l \neq k}^{N} \frac{1}{u_k - u_l}\,, \qquad j = 1, \ldots, N\,.$$

This is a system of N quadratic equations for variables x_j

2^N solutions counted with multiplicities $\{x_1^{(m)}, \ldots, x_N^{(m)}\}$, where $m = 1, \ldots, 2^N$

$$\mathfrak{p}(N) = \sum_{s=N/2-[N/2]}^{N/2} \mathrm{mult}_s(N) = \frac{N!}{(N-[N/2])![N/2]!}$$

number of different tuples $(x_1, \ldots, x_N)$ among all 2^N solutions

Spectral equations for $\mathfrak{gl}_2$ spin chain

$$\mathfrak{p}(N) = \sum_{s=N/2-[N/2]}^{N/2} \text{mult}_s(N) = \frac{N!}{(N-[N/2])![N/2]!}$$

number of different tuples $(x_1,\ldots,x_N)$ among all 2^N solutions

For $N=4$ there will be $\mathfrak{p}(4)=6$ different triples (x_1,x_2,x_2)

Level	Spin pattern	# of states	$\mathfrak{S}_{N=4}$-modules
$M=0$	↑↑↑↑	1	$\mathcal{S}^{[4]}_{s=2}$
$M=1$	↓↑↑↑	4	$\mathcal{S}^{[3,1]}_{s=1} \oplus \mathcal{S}^{[4]}$
$M=2$	↓↓↑↑	6	$\mathcal{S}^{[2,2]}_{s=0} \oplus \mathcal{S}^{[3,1]} \oplus \mathcal{S}^{[4]}$
$M=3$	↓↓↓↑	4	$\mathcal{S}^{[3,1]} \oplus \mathcal{S}^{[4]}$
$M=4$	↓↓↓↓	1	$\mathcal{S}^{[4]}$

$d_{[2,2]}=2,\ d_{[3,1]}=3,\ d_{[4]}=1$

Spectral equations for $\mathfrak{gl}_2$ spin chain

$$t_j = \prod_{k\neq j}^{N} \frac{1}{u_j - u_k}\,, \qquad y_j = x_j t_j$$

$$\sum_{k=1}^{N} \frac{y_j y_k}{u_j - u_k + \eta} + 2y_j + \eta \prod_{k\neq j}^{N} \frac{u_j - u_k - \eta}{u_j - u_k} = 0\,, \quad j = 1, \ldots, N\,.$$

asymptotics of $\mathsf{T}_1(u)$

$$\sum_{j=1}^{N} y_j = -N\eta$$

must be valid for any solution!

$$\sum_{k=1}^{N} \frac{\eta}{u_j - u_k + \eta} \prod_{l\neq j}^{N} \frac{u_k - u_l - \eta}{u_k - u_l} = 1\,, \qquad \forall j = 1, \ldots, N\,.$$

$$\frac{y_j - t_j \prod_{k=1}^{N}(u_j - u_k - \eta)}{y_j} = -\sum_{k=1}^{N} \frac{1}{u_j - u_k + \eta}\left[y_k - t_k \prod_{l=1}^{N}(u_k - u_l - \eta)\right]$$

Introduce

$$\rho_j = \frac{y_j - t_j \prod_{k=1}^{N}(u_j - u_k - \eta)}{y_j}$$

Spectral equations for $\mathfrak{gl}_2$ spin chain

the system takes the form

$$\rho_j - \sum_{k=1}^{N} L_{jk}\rho_k = 0$$

$$K_{jk}\rho_k = 0\,, \qquad j = 1\,,\ldots,N\,,$$

where the $N \times N$ matrix K is

$$K = 1 - L$$

trivial solution $\rho_j = 0$ which for x_j yields

$$x_j = \prod_{k=1}^{N}(u_j - u_k - \eta)$$

$$\mathsf{T}_1^{\mathrm{vac}}(u) = \prod_{j=1}^{N}(u - u_j) + \prod_{j=1}^{N}(u - u_j - \eta) = \mathsf{p}_1(u) + \mathsf{p}_2(u)$$

$s = N/2$ *vacuum multiplet*

For non-trivial solutions $\det K = 0$

Quantum-classical duality

Let us renormalize variables x_j by the vacuum solution $$\zeta_j = \frac{x_j}{\prod_{k=1}^{N}(u_j - u_k - \eta)}$$

$$L = \sum_{i,j=1}^{N} \frac{\eta\,\zeta_j}{u_i - u_j + \eta} \prod_{k\neq j}^{N} \frac{u_j - u_k - \eta}{u_j - u_k} E_{ij}$$

Lax matrix of the Ruijsenaars-Schneider model

$$\zeta_j = e^{-p_j} \qquad \{p_i, u_j\} = \delta_{ij}$$

coordinates

Integrals of motion

$$p_k = \mathrm{Tr}L^k\,, \quad k = 1, \ldots, N\,.$$

Another basis of integrals are ani-symmetric characters $$\det(L - z) = \sum_{k=0}^{N}(-1)^{N-k} z^{N-k} e_k$$

$$\begin{aligned}
e_1 &= \sum_{j=1}^{N} \zeta_j \prod_{k\neq j}^{N} \frac{u_j - u_k - \eta}{u_j - u_k} = \mathrm{tr}L\,,\\
e_2 &= \sum_{i<j}^{N} \zeta_i\zeta_j \prod_{k\neq i,j}^{N} \frac{u_i - u_k - \eta}{u_i - u_k}\frac{u_j - u_k - \eta}{u_j - u_k}\,,\\
&\vdots\\
e_{N-1} &= \sum_{i_1<\ldots<j_{N-1}}^{N} \zeta_{j_1}\ldots\zeta_{j_{N-1}} \prod_{k\neq j_1,\ldots,j_{N-1}}^{N} \frac{u_{j_1} - u_k - \eta}{u_{j_1} - u_k}\cdots\frac{u_{j_{N-1}} - u_k - \eta}{u_{j_{N-1}} - u_k}\,,\\
e_N &= \zeta_1\ldots\zeta_N = \det L\,.
\end{aligned}$$

Theorem. *For any solution of the $\mathfrak{gl}_2$ spectral equations*

$$\mathrm{Tr} L^k = N\,, \quad k = 1, \ldots, N$$

In other words, p_k and e_k are invariants of the spectral equations

$$e_k = C_N^k\,, \quad k = 1, \ldots, N$$

Sketch of the proof

Spectral equations imply recurrence $\mathrm{Tr} L^k = 2\mathrm{Tr} L^{k-1} - \mathrm{Tr} L^{k-2}$ $\longrightarrow$ $\mathrm{Tr} L^k = N + k(\mathrm{Tr} L - N)$

$p_k = \mathrm{Tr} L^k$

$$p_k = N + k(\mu - N)\,, \quad k \in \mathbb{Z}$$

$\mu \equiv p_1 = \mathrm{Tr} L$

$k = -1$ $\longrightarrow$ $\mathrm{Tr} L^{-1} = 2N - \mu = \dfrac{e_{N-1}}{e_N}$

$$e_k = \sum_{n=0}^{k} (-1)^n C_{N+1}^{k-n} L_n(\mu - N) = \sum_{m=0}^{k} \frac{(\mu - N)^m}{m!} C_{N+1}^{k-m} \; {}_2F_1(m+1, m-k; N+2+m-k; 1)$$

$\longrightarrow$

$$e_N = \sum_{m=0}^{N} \frac{(\mu - N)^m}{m!}\,, \qquad e_{N-1} = \sum_{m=0}^{N-1} (N - m) \frac{(\mu - N)^m}{m!}$$

Quantum-classical duality

$$\mathsf{e}_{N-1} + (\mu - 2N)\mathsf{e}_N = 0$$

$$\mathsf{e}_{N-1} + (\mu - 2N)\mathsf{e}_N = \frac{(\mu - N)^{N+1}}{N!} = 0$$

$$\mu = N$$

$$\mathrm{Tr} L^k = N \qquad\qquad \mathsf{e}_k = C_N^k$$

$$0 = \sum_{k=0}^{N} (-1)^{N-k} L^{N-k} \mathsf{e}_k(L) = \sum_{k=0}^{N} C_N^k (-1)^{N-k} L^{N-k} = (1 - L)^N$$

i.e. we deduce that $K = 1 - L$ is nilpotent $\quad K^N = 0$

$$e^t = (1, \ldots, 1)$$

The system of N quadratic equations

$$(1 - L)^2 e = 0$$

is fully equivalent to the original system

Quantum-classical duality

Spectral equations for $\mathfrak{gl}_\ell$ spin chain are equivalent to

$$(1 - L)^\ell e = 0$$

where L is the Lax matrix of the rational RS model for N particles.

Let us fix N and consider equations for $\ell < N$

Denote by $\Omega^N_{\mathfrak{gl}_\ell}$ a set of solutions counted *without* multiplicities

$$\Omega^N_{\mathfrak{gl}_2} \subset \Omega^N_{\mathfrak{gl}_3} \subset \ldots \subset \Omega^N_{\mathfrak{gl}_\ell} \subset \ldots \subset \Omega^N_{\mathfrak{gl}_N}$$

From Schur-Weyl duality, for N fixed the number of $\mathfrak{gl}_\ell$ irreducible multiplets in the tensor product decomposition of $(\mathbb{C}^\ell)^{\otimes N}$ stabilizers starting from $\ell = N$

$$\Omega^N_{\mathfrak{gl}_N} = \Omega^N_{\mathfrak{gl}_{N+1}} = \ldots = \Omega^N_{\mathfrak{gl}_\ell} = \ldots \qquad \ell > N$$

The solution set of N polynomial equations

$$\mathrm{Tr} L^k = N\,, \quad k = 1, \ldots, N\,, \qquad \longleftarrow \quad N! \text{ solutions}$$

exactly coincides with $\Omega^N_{\mathfrak{gl}_N}$ $\longleftarrow$ the number of different solutions among all $N!$ solutions

Analytic Bethe Ansatz

III. Baxter's TQ-relations and wronskian Bethe equations

Tableau sum formula

$$\boxed{j} \quad \leftrightarrow \quad Z_j(u) \qquad 1 \leqslant j \leqslant \ell \text{ for } \mathrm{GL}_\ell$$

quantum eigenvalues

$$Z_i(u) Z_j(v) = Z_j(v) Z_i(u)$$

$$T_\lambda(u) = \sum_{t \in \{\mathsf{asst}_\lambda\}} \prod_{(i,j) \in t} Z_{\#(i,j)}(u + c_{ij}\eta)$$

$c_{ij} = j - i$

content of the (i,j)-box

For instance,

$$T_k(u) = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant \ell} Z_{i_1}(u - (k-1)\eta) \cdots Z_{i_k}(u)$$

compatible with the CBR formula!

Tableau sum formula

$$\mathsf{T}_\lambda(u) = \sum_{t\in\{\mathsf{asst}_\lambda\}} \prod_{(i,j)\in t} \mathsf{Z}_{\#(i,j)}(u + c_{ij}\eta)$$

Example

$$\mathsf{T}_{[2]}(u) = \mathsf{T}_1(u)\,\mathsf{T}_1(u+\eta) - \mathsf{T}_2(u+\eta) \qquad \longleftarrow \qquad \text{CBR formula}$$

$$\mathsf{T}_1(u) = \boxed{1} + \boxed{2} = \mathsf{Z}_1(u) + \mathsf{Z}_2(u)\,,$$

$$\mathsf{T}_2(u) = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} = \mathsf{Z}_1(u-\eta)\mathsf{Z}_2(u)\,,$$

$$\mathsf{T}_{[2]}(u) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} = \mathsf{Z}_1(u)\mathsf{Z}_1(u+\eta) + \mathsf{Z}_1(u+\eta)\mathsf{Z}_2(u) + \mathsf{Z}_2(u)\mathsf{Z}_2(u+\eta)$$

Quantum spectral curve

$$\mathsf{T}_k(u) = \sum_{1\leqslant i_1<\ldots<i_k\leqslant \ell} Z_{i_1}(u)\cdots Z_{i_k}(u+(k-1)\eta)$$

Miura transform

$$\sum_{k=0}^{\ell}(-1)^k \mathsf{T}_k(u) e^{k\eta\partial_u} = (1-Z_1(u)e^{\eta\partial_u})\ldots(1-Z_\ell(u)e^{\eta\partial_u})$$

Recall

$$\det\left(1-\mathsf{M}(u)e^{\eta\partial_u}\right) = \sum_{k=0}^{\ell}(-1)^k \mathsf{T}_k(u)e^{k\eta\partial_u} \equiv L_1(u)$$

finite-difference operator

Classical spectral curve

$$\det(\mathsf{M}(u) - z\mathbb{1}) = 0$$

Quantum spectral curve

Let a function $\mathsf{Q}(u-\eta)$ be in the kernel of $L_1(u)$

$$\sum_{k=0}^{\ell}(-1)^k \mathsf{T}_k(u)\mathsf{Q}(u+(k-1)\eta) = 0$$

Baxter's TQ-relation!

Quantum spectral curve

$$\sum_{k=0}^{\ell}(-1)^k \mathsf{T}_k(u)e^{k\eta\partial_u} = (1 - \mathsf{Z}_1(u)e^{\eta\partial_u})\dots(1 - \mathsf{Z}_\ell(u)e^{\eta\partial_u})$$

$$\mathsf{L}_j = (1 - \mathsf{Z}_j(u)e^{\eta\partial_u})(1 - \mathsf{Z}_{j+1}(u)e^{\eta\partial_u})\dots(1 - \mathsf{Z}_\ell(u)e^{\eta\partial_u})$$

$$\mathrm{Ker}\,\mathsf{L}_\ell(u) \subset \mathrm{Ker}\,\mathsf{L}_{\ell-1}(u) \subset \dots \subset \mathrm{Ker}\,\mathsf{L}_1(u)$$

Let us choose a basis $\omega_k(u)$ of ℓ independent (fundamental) solutions of the ℓth-order difference equation

$$\mathsf{L}_1\omega = 0$$

by requiring that

$$\mathsf{L}_{\ell-j+1}\omega_k = 0\,, \qquad 1 \leqslant k \leqslant j$$

This provides a solution basis compatible with the flag structure

$$\{\omega_1(u)\} \subset \{\omega_1(u), \omega_2(u)\} \subset \dots \subset \{\omega_1(u), \dots, \omega_\ell(u)\}$$

Fundamental Q*-functions* $\mathsf{Q}_j(u) = \omega_j(u+\eta)$

Quantum eigenvalues and characters via Q-functions

$$(1 - Z_\ell(u)e^{\eta\partial_u})\mathsf{Q}_1(u-\eta) = \mathsf{Q}_1(u-\eta) - Z_\ell(u)\mathsf{Q}_1(u) = 0$$

$$\downarrow$$

$$Z_\ell(u) = \frac{\mathsf{Q}_1(u-\eta)}{\mathsf{Q}_1(u)}$$

$$(1 - Z_{\ell-1}(u)e^{\eta\partial_u})(1 - Z_\ell(u)e^{\eta\partial_u})\mathsf{Q}_2(u-\eta) = 0$$

$$\downarrow$$

$$Z_{\ell-1}(u) = \frac{\mathsf{Q}_1(u+\eta)}{\mathsf{Q}_1(u)}\frac{\mathsf{Q}_{\{12\}}(u-\eta)}{\mathsf{Q}_{\{12\}}(u)}$$

$$\mathsf{Q}_{\{1,2\}}(u) = \begin{vmatrix} \mathsf{Q}_1(u) & \mathsf{Q}_1(u+\eta) \\ \mathsf{Q}_2(u) & \mathsf{Q}_2(u+\eta) \end{vmatrix}$$

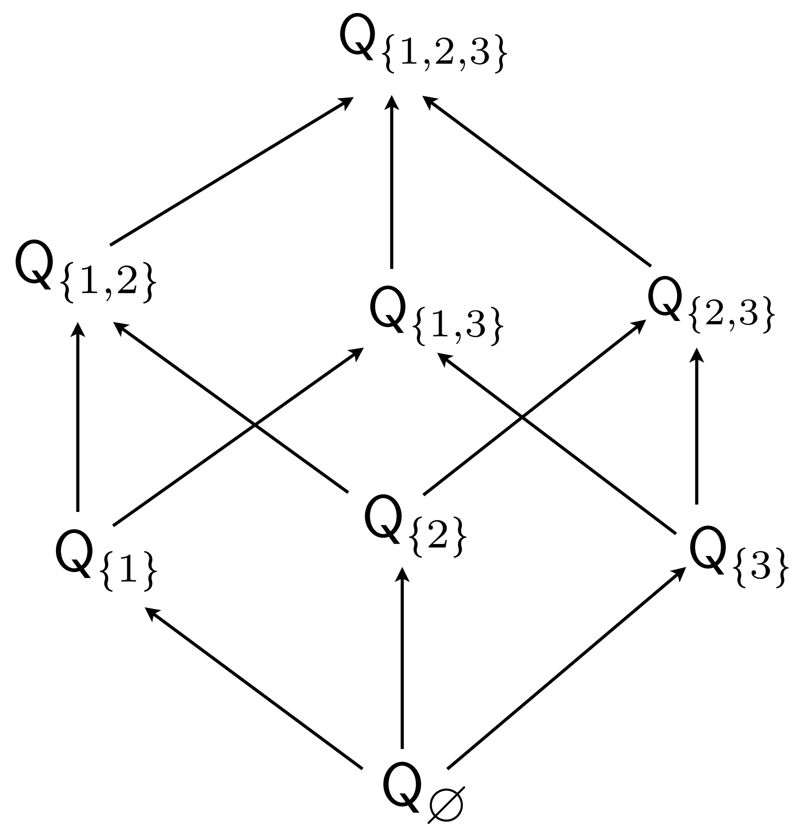
introduce the Q-*functions* $\mathsf{Q}_{\{i_1,\ldots,i_k\}}(u)$

$$\mathsf{Q}_{\{i_1,\ldots,i_k\}}(u) = \begin{vmatrix} \mathsf{Q}_{i_1}(u) & \ldots & \mathsf{Q}_{i_1}(u+(k-1)\eta) \\ \vdots & & \vdots \\ \mathsf{Q}_{i_k}(u) & \ldots & \mathsf{Q}_{i_k}(u+(k-1)\eta) \end{vmatrix}$$

$$\mathsf{Q}_{\{j\}} = \mathsf{Q}_j$$

$$\mathsf{Q}_{\varnothing} = 1$$

subsets of $\{1,\ldots,\ell\}$ $\longrightarrow$ 2^ℓ Q-*functions*



ℓ-dimensional hypercube

Hasse diagram for $\mathfrak{gl}_3$

There are $2^3 = 8$ nodes corresponding to different Q-functions and there are 3!=6 inequivalent paths connecting $Q_{\varnothing}$ with $Q_{\{1,2,3\}}$

$$Z_k(u) = \frac{Q_{\{1,\dots,\ell-(k-1)\}}(u-\eta)}{Q_{\{1,\dots,\ell-(k-1)\}}(u)} \frac{Q_{\{1,\dots,\ell-k\}}(u+\eta)}{Q_{\{1,\dots,\ell-k\}}(u)}$$

$$T_1(u) = \sum_{k=1}^{\ell} \frac{Q_{\{1,\dots,\ell-(k-1)\}}(u-\eta)}{Q_{\{1,\dots,\ell-(k-1)\}}(u)} \frac{Q_{\{1,\dots,\ell-k\}}(u+\eta)}{Q_{\{1,\dots,\ell-k\}}(u)}$$

Solving TQ-relations for $\mathfrak{gl}_2$

$$\mathsf{T}_1(u)\mathsf{Q}(u) = \mathsf{Q}(u-\eta) + \mathsf{T}_2(u)\mathsf{Q}(u+\eta)$$

where $\mathsf{Q} = \mathsf{Q}_1$ or $\mathsf{Q} = \mathsf{Q}_2$

$$\mathsf{T}_2(u) = \prod_{j=1}^{N}(u-u_j+\eta)(u-u_j-\eta)$$

quantum determiant

$$\mathsf{T}_1(u) = \frac{\mathsf{Q}(u-\eta)}{\mathsf{Q}(u)} + \prod_{j=1}^{N}(u-u_j-\eta)\prod_{j=1}^{N}(u-u_j+\eta)\frac{\mathsf{Q}(u+\eta)}{\mathsf{Q}(u)}$$

the asymptotic behavior of transfer matrices at large u

$$\mathsf{T}_1(u) = 2u^N + \dots\,, \qquad \mathsf{T}_2 = u^{2N} + \dots$$

Fusion

$$\mathsf{T}_1(u_j) = \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)} + \prod_{k=1}^{N}(u_j-u_k-\eta)\prod_{k=1}^{N}(u_j-u_k+\eta)\frac{\mathsf{Q}(u_j+\eta)}{\mathsf{Q}(u_j)}\,,$$

$$\mathsf{T}_1(u_j+\eta) = \frac{\mathsf{Q}(u_j)}{\mathsf{Q}(u_j+\eta)}\,.$$

$$\mathsf{T}_1(u_j)\mathsf{T}_1(u_j+\eta) = \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)}\frac{\mathsf{Q}(u_j)}{\mathsf{Q}(u_j+\eta)} + \mathsf{T}_2(u_j) \longrightarrow \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)} = 0$$

Solving TQ-relations for $\mathfrak{gl}_2$

$$\frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)}=0 \qquad\qquad j=1,\ldots,N$$

$$\frac{\mathsf{Q}(u-\eta)}{\mathsf{Q}(u)}=\gamma\prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)}$$

$$\downarrow$$

$$\frac{\mathsf{Q}(u+\eta)}{\mathsf{Q}(u)}=\frac{1}{\gamma\prod_{j=1}^{N}(u-u_j+\eta)}\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)}$$

$$\mathsf{T}_1(u)=\gamma\prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)}+\frac{1}{\gamma}\prod_{j=1}^{N}(u-u_j-\eta)\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)}$$

$$\mathsf{T}_1(u)\sim 2u^N \quad\longrightarrow\quad \gamma+1/\gamma=2 \longrightarrow \gamma=1$$

Analytic Bethe Ansatz

$$\mathsf{T}_1(u) = \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)} + \prod_{j=1}^{N}(u-u_j-\eta)\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)}$$

apparent poles at $u = v_k$

$$\mathcal{Q}(u) = \prod_{j=1}^{M}(u-v_j)$$

the location of its roots v_j can be found from the system of Bethe equations

$$\prod_{j=1}^{N}\frac{v_k-u_j-\eta}{v_k-u_j} = -\frac{\mathcal{Q}(v_k-\eta)}{\mathcal{Q}(v_k+\eta)}\,, \qquad k=1,\ldots,M$$

analyticity of $\mathsf{T}_1(u)$ at $u = v_k$

Baxter Q-polynomial

$$\mathsf{T}_1(u) = \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)} + \prod_{j=1}^{N}(u-u_j-\eta)\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)}$$

$\mathsf{p}_2(u)$ $\quad$ $\mathsf{p}_1(u)$ $\quad$ ← vacuum polynomials

Full agreement with the algebraic Bethe Ansatz!

Analytic Bethe Ansatz

Analytic structure of Q*-functions*

$$\mathsf{Q}(u) = \tau(u)\mathcal{Q}(u) \qquad \mathcal{Q}(u) = \prod_{j=1}^{M}(u - v_j)$$

$$\frac{\mathsf{Q}(u-\eta)}{\mathsf{Q}(u)} = \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)} \longrightarrow \frac{\tau(u-\eta)}{\tau(u)} = \prod_{j=1}^{N}(u-u_j)$$

$$\tau(u) = \sigma(u)\prod_{j=1}^{N}(-\eta)^{-\frac{u}{\eta}}\Gamma\left(\frac{u_j - u}{\eta}\right)$$

$$\sigma(u+\eta) = \sigma(u)$$

$$\mathsf{Q}(u) = \mu\,\mathcal{Q}(u)\prod_{j=1}^{N}(-\eta)^{-\frac{u}{\eta}}\Gamma\left(\frac{u_j - u}{\eta}\right)$$

$\mathsf{Q}_1(u)$ and $\mathsf{Q}_2(u)$ must have the same structure, albeit with different normalization constants μ_1 and μ_2

$$\frac{\mathsf{Q}_1(u-\eta)}{\mathsf{Q}_1(u)} = \prod_{j=1}^{N}(u-u_j)\,\frac{\mathcal{Q}_1(u-\eta)}{\mathcal{Q}_1(u)}\,, \qquad \frac{\mathsf{Q}_2(u-\eta)}{\mathsf{Q}_2(u)} = \prod_{j=1}^{N}(u-u_j)\,\frac{\mathcal{Q}_2(u-\eta)}{\mathcal{Q}_2(u)}$$

Wronskian Bethe equations $\mathfrak{gl}_2$

$$\frac{\mathsf{Q}_1(u-\eta)}{\mathsf{Q}_1(u)} = \prod_{j=1}^{N}(u-u_j)\,\frac{\mathcal{Q}_1(u-\eta)}{\mathcal{Q}_1(u)}\,, \qquad \frac{\mathsf{Q}_2(u-\eta)}{\mathsf{Q}_2(u)} = \prod_{j=1}^{N}(u-u_j)\,\frac{\mathcal{Q}_2(u-\eta)}{\mathcal{Q}_2(u)}$$

Recall the solution for the transfer matrices in terms of quantum eigenvalues, the latter are solved in terms of Q's

$$\mathsf{T}_1(u) = \frac{\mathsf{Q}_1(u-\eta)}{\mathsf{Q}_1(u)} + \frac{\mathsf{Q}_1(u+\eta)}{\mathsf{Q}_1(u)}\,\frac{\mathsf{Q}_{\{1,2\}}(u-\eta)}{\mathsf{Q}_{\{1,2\}}(u)}\,,$$

$$\mathsf{T}_2(u) = \frac{\mathsf{Q}_{\{1,2\}}(u-\eta)}{\mathsf{Q}_{\{1,2\}}(u)}\,.$$

$$\mathsf{T}_2(u) = -\frac{\frac{\mathsf{Q}_1(u-\eta)}{\mathsf{Q}_1(u)} - \frac{\mathsf{Q}_2(u-\eta)}{\mathsf{Q}_2(u)}}{\frac{\mathsf{Q}_1(u+\eta)}{\mathsf{Q}_1(u)} - \frac{\mathsf{Q}_2(u+\eta)}{\mathsf{Q}_2(u)}} \longrightarrow \prod_{j=1}^{N}\frac{u-u_j-\eta}{u-u_j} = -\frac{\mathcal{Q}_1(u-\eta)\mathcal{Q}_2(u) - \mathcal{Q}_2(u-\eta)\mathcal{Q}_1(u)}{\mathcal{Q}_1(u+\eta)\mathcal{Q}_2(u) - \mathcal{Q}_2(u+\eta)\mathcal{Q}_1(u)}$$

Wronskian Bethe equations

$$a\prod_{j=1}^{N}(u-u_j) = \mathcal{Q}_1(u)\mathcal{Q}_2(u+\eta) - \mathcal{Q}_2(u)\mathcal{Q}_1(u+\eta) = \begin{vmatrix} \mathcal{Q}_1(u) & \mathcal{Q}_1(u+\eta) \\ \mathcal{Q}_2(u) & \mathcal{Q}_2(u+\eta) \end{vmatrix}$$

Wronskian Bethe equations $\mathfrak{gl}_2$

$$a\prod_{j=1}^{N}(u-u_j) = \mathcal{Q}_1(u)\mathcal{Q}_2(u+\eta) - \mathcal{Q}_2(u)\mathcal{Q}_1(u+\eta) = \begin{vmatrix} \mathcal{Q}_1(u) & \mathcal{Q}_1(u+\eta) \\ \mathcal{Q}_2(u) & \mathcal{Q}_2(u+\eta) \end{vmatrix}$$

$$\mathcal{Q}_1(u) = \prod_{j=1}^{M}(u-v_j)\,, \qquad \mathcal{Q}_2(u) = \prod_{j=1}^{M^*}(u-v_j^*)$$

$$au^N + \ldots = \eta(M^* - M)u^{M+M^*-1} + \ldots$$

$$M^* = N - M + 1\,, \qquad a = (N-2M+1)\eta \qquad\qquad M^* > [N/2]$$

$$\begin{aligned} \mathsf{T}_1(u) &= \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}_1(u-\eta)\mathcal{Q}_2(u+\eta) - \mathcal{Q}_1(u+\eta)\mathcal{Q}_2(u-\eta)}{\mathcal{Q}_1(u)\mathcal{Q}_2(u+\eta) - \mathcal{Q}_1(u+\eta)\mathcal{Q}_2(u)} \\ &= \frac{1}{(N-2M+1)\eta}\begin{vmatrix} \mathcal{Q}_1(u-\eta) & \mathcal{Q}_1(u+\eta) \\ \mathcal{Q}_2(u-\eta) & \mathcal{Q}_2(u+\eta) \end{vmatrix}. \end{aligned}$$

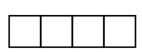
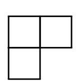
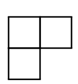
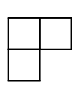
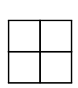
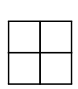
Wronskian Bethe equations $\mathfrak{gl}_2$

$$a\prod_{j=1}^{N}(u-u_j) = \mathcal{Q}_1(u)\mathcal{Q}_2(u+\eta) - \mathcal{Q}_2(u)\mathcal{Q}_1(u+\eta) = \begin{vmatrix} \mathcal{Q}_1(u) & \mathcal{Q}_1(u+\eta) \\ \mathcal{Q}_2(u) & \mathcal{Q}_2(u+\eta) \end{vmatrix}$$

$$\mathcal{Q}_1(u) = u^M\left(1+\sum_{k=1}^{M}\frac{a_1^{(k)}}{u^k}\right), \qquad \mathcal{Q}_2(u) = u^{M^*}\left(1+\sum_{k=1}^{M^*}\frac{a_2^{(k)}}{u^k}\right)$$

$$\mathcal{Q}_2(u) \to \mathcal{Q}_2(u) + \alpha\,\mathcal{Q}_1(u)$$

$\to$ N polynomial equations to determine all the remaining $N = M + (M^* - 1)$ coefficients

Tableau	Polynomials $\mathcal{Q}_1(u), \mathcal{Q}_2(u)$	$\mathsf{T}_1(u)$
	$1, u^5 - \frac{5}{2}u^4\eta + \frac{5}{3}u^3\eta^2 - \frac{1}{6}u\eta^4$	$2u^4 - 4u^3\eta + 6u^2\eta^2 - 4u\eta^3 + \eta^4$
	$u - \frac{1}{2}\eta\,, u^4 - 2u^3\eta + 2u^2\eta^2 - \frac{1}{2}\eta^4$	$2u^4 - 4u^3\eta + 2u^2\eta^2 - \eta^4$
	$u - \frac{1+i}{2}\eta, u^4 - (2-i)u^3\eta + \frac{1-3i}{2}u^3\eta + \frac{i}{2}\eta^4$	$2u^4 - 4u^3\eta + 2u^2\eta^2 - 2iu\eta^3 + i\eta^4$
	$u - \frac{1-i}{2}\eta, u^4 - (2+i)u^3\eta + \frac{1+3i}{2}u^3\eta - \frac{i}{2}\eta^4$	$2u^4 - 4u^3\eta + 2u^2\eta^2 + 2iu\eta^3 - i\eta^4$
	$u^2 - u\eta, u^3 - 2u^2\eta + \frac{1}{2}\eta^3$	$2u^4 - 4u^3\eta + 2u\eta^3 - \eta^4$
	$u^2 - u\eta + \frac{1}{2}\eta^3, u^3 - u^2\eta + \frac{1}{6}\eta^3$	$2u^4 - 4u^3\eta + 2u\eta^3 + \eta^4$

Representation content, $\mathcal{Q}$-polynomials and transfer matrix eigenvalues $\mathsf{T}_1(u)$ for the $\mathfrak{gl}_2$ spin chain of length $N = 4$. The value of the quantum determinant is $\mathsf{T}_2(u) = (u-\eta)^4(u+\eta)^4$

TQ-relations for $\mathfrak{gl}_\ell$

$$\mathsf{T}_1(u) = \frac{\mathsf{Q}(u-\eta)}{\mathsf{Q}(u)} + \mathsf{T}_2(u)\frac{\mathsf{Q}(u+\eta)}{\mathsf{Q}(u)} + \ldots + (-1)^\ell \mathsf{T}_\ell(u)\frac{\mathsf{Q}(u+(\ell-1)\eta)}{\mathsf{Q}(u)}$$

$$\mathsf{T}_1(u_j+\eta) = \frac{\mathsf{Q}(u_j)}{\mathsf{Q}(u_j+\eta)} + \mathsf{T}_2(u_j+\eta)\frac{\mathsf{Q}(u_j+2\eta)}{\mathsf{Q}(u_j+\eta)} + \ldots + (-1)^\ell \mathsf{T}_\ell(u_j+\eta)\frac{\mathsf{Q}(u_j+\ell\eta)}{\mathsf{Q}(u_j+\eta)}$$

$\mathsf{T}_\ell(u_j+\eta) = 0$

$$\mathsf{T}_2(u_j) = \mathsf{T}_1(u_j)\mathsf{T}_1(u_j+\eta)$$
$$= \mathsf{T}_1(u_j)\frac{\mathsf{Q}(u_j)}{\mathsf{Q}(u_j+\eta)} + \mathsf{T}_3(u_j)\frac{\mathsf{Q}(u_j+2\eta)}{\mathsf{Q}(u_j+\eta)} + \ldots + (-1)^{\ell-1} \mathsf{T}_\ell(u_j)\frac{\mathsf{Q}(u_j+(\ell-1)\eta)}{\mathsf{Q}(u_j+\eta)}$$

substitute

$$\mathsf{T}_1(u_j) = \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)} + \mathsf{T}_2(u_j)\frac{\mathsf{Q}(u_j+\eta)}{\mathsf{Q}(u_j)} + \ldots + (-1)^\ell \mathsf{T}_\ell(u_j)\frac{\mathsf{Q}(u_j+(\ell-1)\eta)}{\mathsf{Q}(u_j)}$$

all terms with $\mathsf{T}_{k\geq 3}$ *cancel out*

$$\mathsf{T}_2(u_j) = \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)}\frac{\mathsf{Q}(u_j)}{\mathsf{Q}(u_j+\eta)} + \mathsf{T}_2(u_j) \longrightarrow \frac{\mathsf{Q}(u_j-\eta)}{\mathsf{Q}(u_j)} = 0\,, \qquad j=1,\ldots,N$$

TQ-relations for $\mathfrak{gl}_\ell$

$$\frac{\mathsf{Q}(u-\eta)}{\mathsf{Q}(u)} = \gamma \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)}$$

↓

$$\frac{\mathsf{Q}(u+k\eta)}{\mathsf{Q}(u)} = \frac{1}{\gamma^k g_k(u)}\frac{\mathcal{Q}(u+k\eta)}{\mathcal{Q}(u)}$$

$$g_k(u) = \prod_{j=1}^{N}\prod_{m=1}^{k-1}(u-u_j+m\eta)$$

$$\mathsf{T}_1(u) = \gamma \prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)} + \gamma^{-1}\frac{\mathsf{T}_2(u)}{g_2(u)}\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)} - \gamma^{-2}\frac{\mathsf{T}_3(u)}{g_3(u)}\frac{\mathcal{Q}(u+2\eta)}{\mathcal{Q}(u)} + \ldots + (-1)^{\ell}\gamma^{-(\ell-1)}\frac{\mathsf{T}_\ell(u)}{g_\ell(u)}\frac{\mathcal{Q}(u+(\ell-1)\eta)}{\mathcal{Q}(u)}$$

$$\ell = \gamma + \sum_{k=2}^{\ell}(-1)^k\gamma^{-(k-1)}C_\ell^k = \ell + \gamma\left(1-\frac{1}{\gamma}\right)^{\ell} \longrightarrow \gamma = 1$$

Connection to the algebraic Bethe Ansatz for $\mathfrak{gl}_\ell$

$$\frac{\mathsf{Q}_{\{1,\ldots,k\}}(u-\eta)}{\mathsf{Q}_{\{1,\ldots,k\}}(u)} = g_k(u)\prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}_{\{1,\ldots,k\}}(u-\eta)}{\mathcal{Q}_{\{1,\ldots,k\}}(u)}$$

$$\frac{\mathsf{Q}_{\{1,\ldots,k\}}(u+\eta)}{\mathsf{Q}_{\{1,\ldots,k\}}(u)} = \frac{1}{g_{k+1}(u)}\frac{\mathcal{Q}_{\{1,\ldots,k\}}(u+\eta)}{\mathcal{Q}_{\{1,\ldots,k\}}(u)},$$

$$\downarrow$$

$$\mathsf{T}_1(u) = \prod_{j=1}^{N}(u-u_j)\sum_{k=1}^{\ell}\frac{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u-\eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u)}\frac{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u+\eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u)}$$

$$\mathsf{T}_\ell(u) = g_\ell(u)\prod_{j=1}^{N}(u-u_j)\frac{\mathcal{Q}_{\{1,\ldots,\ell\}}(u-\eta)}{\mathcal{Q}_{\{1,\ldots,\ell\}}(u)}$$

$$\downarrow$$

$$\frac{\mathcal{Q}_{\{1,\ldots,\ell\}}(u-\eta)}{\mathcal{Q}_{\{1,\ldots,\ell\}}(u)} = \prod_{j=1}^{N}\frac{u-u_j-\eta}{u-u_j}$$

$$\downarrow$$

$$\mathsf{T}_1(u) = \mathsf{p}_1(u)\frac{\mathcal{Q}_{\{1,\ldots,\ell-1\}}(u+\eta)}{\mathcal{Q}_{\{1,\ldots,\ell-1\}}(u)} + \sum_{k=2}^{\ell}\mathsf{p}_k(u)\frac{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u-\eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u)}\frac{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u+\eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u)}$$

Reproduction of the algebraic Bethe Ansatz result!

Traditional $\mathfrak{gl}_\ell$ Bethe equations

$$\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u) = \varkappa_{\ell-k} \prod_{j=1}^{\boldsymbol{M}_k} \left(u - u_j^{(k)}\right)$$

$\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u) = \varkappa_{\ell-k} Q_k(u)$ $\qquad$ $Q_k(u)$ *are Baxter's polynomials*

$$\frac{\mathcal{Q}_{\{1,\ldots,\ell-k-1\}}(u_j^{(k)})}{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u_j^{(k)})} \frac{\mathcal{Q}_{\{1,\ldots,\ell-k+1\}}(u_j^{(k)} - \eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k-1\}}(u_j^{(k)} + \eta)} = - \frac{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u_j^{(k)} - \eta)}{\mathcal{Q}_{\{1,\ldots,\ell-k\}}(u_j^{(k)} + \eta)}$$

where $j = 1, \ldots, \boldsymbol{M}_k$ and $k = 1, \ldots, \ell - 1$

traditional Bethe equations

Wronskian Bethe equations $\mathfrak{gl}_\ell$

$$\frac{\mathcal{Q}_{\{1,\dots,\ell\}}(u)}{\prod_{j=1}^{N}(u-u_j)} = \frac{\mathcal{Q}_{\{1,\dots,\ell\}}(u-\eta)}{\prod_{j=1}^{N}(u-u_j-\eta)}$$

$$f(u) = \frac{\mathcal{Q}_{\{1,\dots,\ell\}}(u)}{\prod_{j=1}^{N}(u-u_j)} \longrightarrow f(u) = f(u+\eta) \longrightarrow f(u) = a$$

Quantization condition!

$$\mathcal{Q}_{\{1,\dots,\ell\}}(u) = a\prod_{j=1}^{N}(u-u_j)$$

$$\downarrow$$

$$\begin{vmatrix} \mathcal{Q}_1(u) & \dots & \mathcal{Q}_1(u+(\ell-1)\eta) \\ \vdots & & \vdots \\ \mathcal{Q}_\ell(u) & \dots & \mathcal{Q}_\ell(u+(\ell-1)\eta) \end{vmatrix} = \varkappa_\ell \prod_{j=1}^{N}(u-u_j)$$

$$\mathsf{T}_k(u) = \frac{g_k(u)}{\varkappa_\ell}\det\Big[\mathcal{Q}_i\big(u+(j-1-\delta_{j|k})\eta\big)\Big]_{1\leqslant i,j\leqslant \ell}$$

Wronskian Bethe equations $\mathfrak{gl}_\ell$

$$\begin{vmatrix} \mathcal{Q}_1(u) & \dots & \mathcal{Q}_1(u+(\ell-1)\eta) \\ \vdots & & \vdots \\ \mathcal{Q}_\ell(u) & \dots & \mathcal{Q}_\ell(u+(\ell-1)\eta) \end{vmatrix} = \varkappa_\ell \prod_{j=1}^{N} (u-u_j)$$

and from matching the leading power of u in

$$\mathcal{Q}_k(u) = \prod_{s=1}^{n_k} \left(u - w_s^{(k)}\right) = u^{n_k} \left(1 + \sum_{s=1}^{n_k} \frac{a_k^{(s)}}{u^s}\right)$$

wronskian Bethe roots

$$\mathcal{Q}_{\{1,2,\dots,k\}}(u) = \varkappa_k u^{\nu_k} + \dots,$$

compare

$$\nu_k = \sum_{j=1}^{k} n_j - \frac{k(k-1)}{2}, \qquad \varkappa_k = \eta^{\frac{k(k-1)}{2}} \prod_{i>j}^{k} (n_i - n_j), \qquad k = 2, \dots, \ell$$

$$\nu_1 = n_1 \text{ and } \varkappa_1 = 1$$

$$\mathcal{Q}_{\{1,\dots,\ell-k\}}(u) = \varkappa_{\ell-k} \prod_{j=1}^{\boldsymbol{M}_k} \left(u - u_j^{(k)}\right)$$

# of traditional Bethe roots

$$\boldsymbol{M}_k = \sum_{j=1}^{\ell-k} n_j - \frac{(\ell-k)(\ell-1-k)}{2}, \qquad k = 1, \dots, \ell-1,$$

and from matching the leading power of u

$$N = \sum_{j=1}^{\ell} n_j - \frac{\ell(\ell-1)}{2}$$

Wronskian Bethe equations $\mathfrak{gl}_\ell$

$$n_k = \boldsymbol{M}_{\ell-k} - \boldsymbol{M}_{\ell-k+1} + k - 1$$

$[M_1, \ldots, M_\ell]$ $\mathfrak{gl}_\ell$ *highest weight*

$$n_k = M_{\ell-k+1} + k - 1$$

$$n_1 = M_\ell\,, \quad n_2 = M_{\ell-1} + 1\,, \quad n_2 = M_{\ell-2} + 2\,, \quad \ldots \quad , n_\ell = M_1 + \ell - 1$$

$$M_1 \geqslant M_2 \geqslant \ldots \geqslant M_\ell$$

$$n_1 < n_2 < \ldots < n_\ell$$

together with the restriction

$$\sum_{k=1}^{\ell} n_k = N + \frac{\ell(\ell-1)}{2}$$

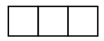
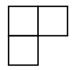
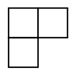
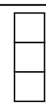
number of unknowns

Gauge transformations

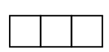
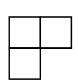
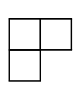
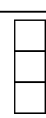
$$\begin{aligned}
\mathcal{Q}_1(u) &\to \mathcal{Q}_1(u)\,,\\
\mathcal{Q}_2(u) &\to \mathcal{Q}_2(u) + \alpha_{21}\mathcal{Q}_1(u)\,,\\
\mathcal{Q}_3(u) &\to \mathcal{Q}_3(u) + \alpha_{32}\mathcal{Q}_2(u) + \alpha_{31}\mathcal{Q}_1(u)\,,\\
&\vdots\\
\mathcal{Q}_\ell(u) &\to \mathcal{Q}_\ell(u) + \alpha_{\ell\ell-1}\mathcal{Q}_{\ell-1}(u) + \alpha_{\ell\ell-2}\mathcal{Q}_{\ell-2}(u) + \ldots + \alpha_{\ell 1}\mathcal{Q}_1(u)
\end{aligned}$$

$\frac{1}{2}\ell(\ell-1)$ arbitrary constants α_{km}

Example: $\mathfrak{gl}_3$ chain of length N=3

Tableau	$[M_1, M_2, M_3]_D$	$[n_1, n_2, n_3]$	Polynomials $\mathcal{Q}_1(u), \mathcal{Q}_2(u), \mathcal{Q}_3(u)$
	$[3,0,0]_{10}$	$[0,1,5]$	$1, u, u^5 - 5u^4\eta + \frac{25}{3}u^3\eta^2 - 5u^2\eta^3$
	$[2,1,0]_8$	$[0,2,4]$	$1, u^2 - (2 + \frac{i}{\sqrt{3}})u\eta, u^4 - (4 - \frac{2i}{\sqrt{3}})u^3\eta + (10 - i\sqrt{3})u\eta^3$
	$[2,1,0]_8$	$[0,2,4]$	$1, u^2 - (2 - \frac{i}{\sqrt{3}})u\eta, u^4 - (4 + \frac{2i}{\sqrt{3}})u^3\eta + (10 + i\sqrt{3})u\eta^3$
	$[1,1,1]_1$	$[1,2,3]$	$u - \eta, u^2 - \frac{4}{3}\eta^2, u^3 - 2\eta^3$

Representation content and fundamental $\mathcal{Q}$-polynomials for the $\mathfrak{gl}_3$ spin chain of length $N = 3$. The subscript D in $[M_1, M_2, M_3]_D$ indicates the dimension of the corresponding $\mathfrak{gl}_3$ module.

Tableau	$\mathsf{T}_1(u)$	$\frac{\mathsf{T}_2(u)}{g_2(u)}$	$\frac{\mathsf{T}_3(u)}{g_3(u)}$
	$3u^3 - 3u^2\eta + 3u\eta^2 - \eta^3$	$3u^3 - 6u^2\eta + 6u\eta^2 - 2\eta^3$	$(u-\eta)^3$
	$3u^3 - 3u^2\eta + \frac{1-i\sqrt{3}}{2}\eta^3$	$3u^3 - 6u^2\eta + 3u\eta^2 - \frac{1+i\sqrt{3}}{2}\eta^3$	$(u-\eta)^3$
	$3u^3 - 3u^2\eta + \frac{1+i\sqrt{3}}{2}\eta^3$	$3u^3 - 6u^2\eta + 3u\eta^2 - \frac{1-i\sqrt{3}}{2}\eta^3$	$(u-\eta)^3$
	$3u^3 - 3u^2\eta - 3u\eta^2 - \eta^3$	$3u^3 - 6u^2\eta + 4\eta^3$	$(u-\eta)^3$

$$g_2(u) = (u+\eta)^3 \text{ and } g_3(u) = (u+\eta)^3(u+2\eta)^3$$

$$\mathsf{T}_1(u) = u^3\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)} + \frac{\mathsf{T}_2(u)}{g_2(u)}\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)} - (u-\eta)^3\frac{\mathcal{Q}(u+2\eta)}{\mathcal{Q}(u)}$$