$$\hat{W}^{(1)}_{-n} = -k! \cdot \sum_{m=1}^{n} \sum_{\sum_{i=1}^{m} a_i = -n} : \prod_{j=1}^{m} \frac{p_{a_j}}{a_j} : \left(\sum_{\substack{k_1,\ldots,k_m>0 \\ \sum_{i=1}^{n} k_i = n+1}} \prod_{i=1}^{m} \binom{-a_i}{k_i} \right) \tag{1}$$

where the sum is over all tuples $a = \{a_1, a_2 \ldots a_n\}$ and $\{k_1, \ldots k_n\}$ with the respective conditions, and

$$p_{-k} = k \frac{\partial}{\partial p_k} \tag{2}$$

Generating elements are Ψ_i, F_i, E_i, $i \in \mathbb{Z}_{\geq 0}$ so that

$$\begin{aligned}
[\hat{\Psi}_j, \hat{\Psi}_k] &= 0 \\
[\hat{E}_j, \hat{F}_k] &= \hat{\Psi}_{j+k} \\
[\hat{\Psi}_0, \hat{E}_j] &= 0, \qquad [\hat{\Psi}_0, \hat{F}_j] = 0 \\
[\hat{\Psi}_1, \hat{E}_j] &= 0, \qquad [\hat{\Psi}_1, \hat{F}_j] = 0 \\
[\hat{\Psi}_2, \hat{E}_j] &= 2\hat{E}_j, \qquad [\hat{\Psi}_2, \hat{F}_j] = -2\hat{F}_j
\end{aligned} \tag{3}$$

with additional relations on them:

quadratic

$$
\begin{aligned}
[\hat{E}_{j+3}, \hat{E}_k] - 3[\hat{E}_{j+2}, \hat{E}_{k+1}] + 3[\hat{E}_{j+1}, \hat{E}_{k+2}] - [\hat{E}_j, \hat{E}_{k+3}] - [\hat{E}_{j+1}, \hat{E}_k] + [\hat{E}_j, \hat{E}_{k+1}] &= 0 \\
[\hat{F}_{j+3}, \hat{F}_k] - 3[\hat{F}_{j+2}, \hat{F}_{k+1}] + 3[\hat{F}_{j+1}, \hat{F}_{k+2}] - [\hat{F}_j, \hat{F}_{k+3}] - [\hat{F}_{j+1}, \hat{F}_k] + [\hat{F}_j, \hat{F}_{k+1}] &= 0 \\
[\hat{\Psi}_{j+3}, \hat{E}_k] - 3[\hat{\Psi}_{j+2}, \hat{E}_{k+1}] + 3[\hat{\Psi}_{j+1}, \hat{E}_{k+2}] - [\hat{\Psi}_j, \hat{E}_{k+3}] - [\hat{\Psi}_{j+1}, \hat{E}_k] + [\hat{\Psi}_j, \hat{E}_{k+1}] &= 0 \\
[\hat{\Psi}_{j+3}, \hat{F}_k] - 3[\hat{\Psi}_{j+2}, \hat{F}_{k+1}] + 3[\hat{\Psi}_{j+1}, \hat{F}_{k+2}] - [\hat{\Psi}_j, \hat{F}_{k+3}] - [\hat{\Psi}_{j+1}, \hat{F}_k] + [\hat{\Psi}_j, \hat{F}_{k+1}] &= 0 \qquad (4)
\end{aligned}
$$

and cubic (the Serre relations)

$$
\mathrm{Sym}_{i,j,k}[\hat{E}_i, [\hat{E}_j, \hat{E}_{k+1}]] = 0 \qquad \mathrm{Sym}_{i,j,k}[\hat{F}_i, [\hat{F}_j, \hat{F}_{k+1}]] = 0 \qquad (5)
$$

where the symbol $\mathrm{Sym}_{i,j,k}$ means the symmetrization over the three indices i, j, k.
The shift invariance of the Serre relations: $\mathrm{Sym}_{m+i,m+j,m+k} = 0$ holds along with $\mathrm{Sym}_{i,j,k}$ for any m.
This means that any i, j, k-dependent symbolic corollary of $\mathrm{Sym}_{i,j,k} = 0$ is automatically true for $m + i, m + j, m + k$.

$\hat{\Psi}_0$ commutes with all elements of algebra, it is just the central charge of the algebra:

$$\hat{\Psi}_0 = c \tag{6}$$

The whole algebra is generated by three operators $\hat{\Psi}_3$, $\hat{E}_0$ and $\hat{F}_0$ ($\hat{\Psi}_3 = 6\hat{W}_0$) recursively as

$$\begin{aligned} \hat{E}_{k+1} &= \frac{1}{6}[\hat{\Psi}_3, \hat{E}_k] \\ \hat{F}_{k+1} &= -\frac{1}{6}[\hat{\Psi}_3, \hat{F}_k] \end{aligned} \tag{7}$$

The affine Yangian of $\mathfrak{gl}_1$ (isomorphic to the algebra SH^c, O. Schiffmann and E. Vasserot) is defined to be an associative algebra with quadratic relations

$$\begin{aligned}
[\hat{E}_{j+3},\hat{E}_k]-3[\hat{E}_{j+2},\hat{E}_{k+1}]+3[\hat{E}_{j+1},\hat{E}_{k+2}]-[\hat{E}_j,\hat{E}_{k+3}]-[\hat{E}_{j+1},\hat{E}_k]+[\hat{E}_j,\hat{E}_{k+1}]-&\\
-\beta(\beta-1)\left(\{\hat{E}_j,\hat{E}_k\}+[\hat{E}_{j+1},\hat{E}_k]-[\hat{E}_j,\hat{E}_{k+1}]\right) &= 0\\
[\hat{F}_{j+3},\hat{F}_k]-3[\hat{F}_{j+2},\hat{F}_{k+1}]+3[\hat{F}_{j+1},\hat{F}_{k+2}]-[\hat{F}_j,\hat{F}_{k+3}]-[\hat{F}_{j+1},\hat{F}_k]+[\hat{F}_j,\hat{F}_{k+1}]-&\\
-\beta(\beta-1)\left(\{\hat{F}_j,\hat{F}_k\}+[\hat{F}_{j+1},\hat{F}_k]-[\hat{F}_j,\hat{F}_{k+1}]\right) &= 0\\
[\hat{\Psi}_{j+3},\hat{E}_k]-3[\hat{\Psi}_{j+2},\hat{E}_{k+1}]+3[\hat{\Psi}_{j+1},\hat{E}_{k+2}]-[\hat{\Psi}_j,\hat{E}_{k+3}]-[\hat{\Psi}_{j+1},\hat{E}_k]+[\hat{\Psi}_j,\hat{E}_{k+1}]-&\\
-\beta(\beta-1)\left(\{\hat{\Psi}_j,\hat{E}_k\}+[\hat{\Psi}_{j+1},\hat{E}_k]-[\hat{\Psi}_j,\hat{E}_{k+1}]\right) &= 0\\
[\hat{\Psi}_{j+3},\hat{F}_k]-3[\hat{\Psi}_{j+2},\hat{F}_{k+1}]+3[\hat{\Psi}_{j+1},\hat{F}_{k+2}]-[\hat{\Psi}_j,\hat{F}_{k+3}]-[\hat{\Psi}_{j+1},\hat{F}_k]+[\hat{\Psi}_j,\hat{F}_{k+1}]-&\\
-\beta(\beta-1)\left(\{\hat{\Psi}_j,\hat{F}_k\}+[\hat{\Psi}_{j+1},\hat{F}_k]-[\hat{\Psi}_j,\hat{F}_{k+1}]\right) &= 0 \quad (8)
\end{aligned}$$

instead of (4), while all other relations do not change. Here $\{\ldots\}$ denotes the anticommutator, and β is some deformation constant.

The most general deformation depends on two parameters σ_2 and σ_3, which are related with β by $\sigma_2 = -1 - \beta(\beta - 1)$, $\sigma_3 = -\beta(\beta - 1)$. In another parametrization, $\sigma_1 = h_1 + h_2 + h_3 = 0$, $\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3$, $\sigma_3 = h_1 h_2 h_3$, the relation is $h_1 = 1$, $h_2 = -\beta$, $h_3 = \beta - 1$.
One again generates the whole algebra starting from the three generating elements $\hat{\Psi}_3$, $\hat{E}_0$ and $\hat{F}_0$ (this time $\hat{\Psi}_3 - \beta(\beta-1)\hat{\Psi}_2 = 6\hat{W}_0$), and

$$\begin{aligned} \hat{E}_{k+1} &= \frac{1}{6}[\hat{\Psi}_3, \hat{E}_k] - \frac{c}{3}\beta(\beta-1)\hat{E}_k \\ \hat{F}_{k+1} &= -\frac{1}{6}[\hat{\Psi}_3, \hat{F}_k] + \frac{c}{3}\beta(\beta-1)\hat{F}_k \end{aligned} \quad (9)$$