Internal Lagrangians of PDEs as variational principles

Kostya Druzhkov

RDP School and Workshop on Mathematical Physics,
August 19-24, 2023, Yerevan

Outline

1. Main problems
   - Why (how) is the Lagrangian formalism encoded in the intrinsic geometry of a variational equation?
   - What is known to the intrinsic geometry of a differential equation about its variational nature?
2. Main results
   - The notion of stationary point of an internal Lagrangian.
   - The notion of ξ-stationaty point of an internal Lagrangian.

Basic notation

Let us consider a locally trivial smooth vector bundle $\pi\colon E \to M$. Here

- $\dim M = n$, $\dim E = n + m$;
- $x^1, \ldots, x^n$ are local coordinates in $U \subset M$ (independent variables);
- $u^1, \ldots, u^m$ are local coordinates along the fibres of π over U.

The bundle π determines the corresponding bundle of infinite jets

$$\pi_\infty\colon J^\infty(\pi) \to M$$

with the adapted local coordinates u^i_α along the fibers. The Cartan distribution is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+x^k}\partial_{u^i_\alpha}\,, \qquad k = 1, \ldots, n,\ |\alpha| \geqslant 0.$$

Here α is a multi-index of the form $\alpha = \alpha_i x^i$ (just a formal linear combination), $\alpha_i \geqslant 0$; $\quad |\alpha| = \alpha_1 + \ldots + \alpha_n$.

Basic notation

Let us consider a locally trivial smooth vector bundle $\pi\colon E \to M$. Here

- $\dim M = n$, $\dim E = n + m$;
- $x^1, \dots, x^n$ are local coordinates in $U \subset M$ (independent variables);
- $u^1, \dots, u^m$ are local coordinates along the fibres of π over U.

The bundle π determines the corresponding bundle of infinite jets

$$\pi_\infty\colon J^\infty(\pi) \to M$$

with the adapted local coordinates u^i_α along the fibers. The Cartan distribution is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+x^k}\partial_{u^i_\alpha}\,, \qquad k = 1, \dots, n,\ |\alpha| \geqslant 0.$$

Here α is a multi-index of the form $\alpha = \alpha_i x^i$ (just a formal linear combination), $\alpha_i \geqslant 0$; $\quad |\alpha| = \alpha_1 + \ldots + \alpha_n$.

Basic notation

By a Cartan differential form we mean a form vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i \,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{1}$$

in adapted local coordinates. The module $\mathcal{C}\Lambda^1(\pi)$ determines the corresponding ideal of the algebra $\Lambda^*(\pi)$. We denote by $\mathcal{C}^2\Lambda^*(\pi)$ the wedge square of this ideal.

Horizontal n-forms can be regarded as Lagrangians

$$\Lambda_h^n(\pi) = \Lambda^n(\pi)/\mathcal{C}\Lambda^n(\pi)\,. \tag{2}$$

If $L \in \Lambda^n(\pi)$ has the form $L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$, then $\mathrm{E}[L]_h$ is defined by

$$\mathrm{E}[L]_h = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial\lambda}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3}$$

Basic notation

By a Cartan differential form we mean a form vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i\,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{1}$$

in adapted local coordinates. The module $\mathcal{C}\Lambda^1(\pi)$ determines the corresponding ideal of the algebra $\Lambda^*(\pi)$. We denote by $\mathcal{C}^2\Lambda^*(\pi)$ the wedge square of this ideal.

Horizontal n-forms can be regarded as Lagrangians

$$\Lambda_h^n(\pi) = \Lambda^n(\pi)/\mathcal{C}\Lambda^n(\pi)\,. \tag{2}$$

If $L \in \Lambda^n(\pi)$ has the form $L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$, then $\mathrm{E}[L]_h$ is defined by

$$\mathrm{E}[L]_h = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial \lambda}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3}$$

Basic notation

By a Cartan differential form we mean a form vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i \,, \qquad \theta_\alpha^i = du_\alpha^i - u^i_{\alpha+x^k} dx^k \tag{1}$$

in adapted local coordinates. The module $\mathcal{C}\Lambda^1(\pi)$ determines the corresponding ideal of the algebra $\Lambda^*(\pi)$. We denote by $\mathcal{C}^2\Lambda^*(\pi)$ the wedge square of this ideal.

Horizontal n-forms can be regarded as Lagrangians

$$\Lambda_h^n(\pi) = \Lambda^n(\pi)/\mathcal{C}\Lambda^n(\pi)\,. \tag{2}$$

If $L \in \Lambda^n(\pi)$ has the form $L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$, then $\mathrm{E}[L]_h$ is defined by

$$\mathrm{E}[L]_h = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial \lambda}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3}$$

Basic notation

By a Cartan differential form we mean a form vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i \,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{1}$$

in adapted local coordinates. The module $\mathcal{C}\Lambda^1(\pi)$ determines the corresponding ideal of the algebra $\Lambda^*(\pi)$. We denote by $\mathcal{C}^2\Lambda^*(\pi)$ the wedge square of this ideal.

Horizontal n-forms can be regarded as Lagrangians

$$\Lambda_h^n(\pi) = \Lambda^n(\pi)/\mathcal{C}\Lambda^n(\pi)\,. \tag{2}$$

If $L \in \Lambda^n(\pi)$ has the form $L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$, then $\mathrm{E}[L]_h$ is defined by

$$\mathrm{E}[L]_h = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial \lambda}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3}$$

Internal Lagrangians

Suppose we have a system of differential equations

$$F = 0\,, \tag{4}$$

where F is a section of some bundle of the form $\pi_\infty^*(\eta)$. Denote by $\mathcal{E}$ its infinite prolongation. Let

$$L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$$

be an n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ determines a unique element of the quotient

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))}\,, \tag{5}$$

while the cohomology class $[L]_h + d_h\,\Lambda_h^{n-1}(\pi)$ defines a unique element of

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))}\,. \tag{6}$$

We refer to elements of (6) as internal Lagrangians of $\mathcal{E}$.

Internal Lagrangians

Suppose we have a system of differential equations

$$F = 0\,, \tag{4}$$

where F is a section of some bundle of the form $\pi_\infty^*(\eta)$. Denote by $\mathcal{E}$ its infinite prolongation. Let

$$L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$$

be an n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ determines a unique element of the quotient

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))}\,, \tag{5}$$

while the cohomology class $[L]_h + d_h\,\Lambda_h^{n-1}(\pi)$ defines a unique element of

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))}\,. \tag{6}$$

We refer to elements of (6) as internal Lagrangians of $\mathcal{E}$.

Internal Lagrangians

Suppose we have a system of differential equations

$$F = 0\,, \tag{4}$$

where F is a section of some bundle of the form $\pi_\infty^*(\eta)$. Denote by $\mathcal{E}$ its infinite prolongation. Let

$$L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$$

be an n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ determines a unique element of the quotient

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))}\,, \tag{5}$$

while the cohomology class $[L]_h + d_h\,\Lambda_h^{n-1}(\pi)$ defines a unique element of

$$\frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))}\,. \tag{6}$$

We refer to elements of (6) as internal Lagrangians of $\mathcal{E}$.

Internal Lagrangians

Elements of quotient (5) can be treated as integral functionals. To this end, we fix a compact oriented n-dimensional manifold N with boundary ∂N and introduce two notions of "almost solutions of $\mathcal{E}$".

Definition

An embedding $\sigma\colon N \to \mathcal{E}$ is an *almost Cartan embedding* if

$$\dim\left(T_p\, \sigma(N) \cap \mathcal{C}_p\right) \geqslant n - 1 \qquad \text{for all} \quad p \in \sigma(N)\,.$$

Notation: $\sigma \in \mathcal{A}_N(\mathcal{E})$.

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ *defines a boundary value problem* if

$$T_p\, \sigma(\partial N) \subset \mathcal{C}_p \qquad \text{for all} \quad p \in \sigma(\partial N)\,.$$

Notation: $\sigma \in \mathcal{BA}_N(\mathcal{E})$.

Internal Lagrangians

Elements of quotient (5) can be treated as integral functionals. To this end, we fix a compact oriented n-dimensional manifold N with boundary ∂N and introduce two notions of "almost solutions of $\mathcal{E}$".

Definition

An embedding $\sigma\colon N \to \mathcal{E}$ is an *almost Cartan embedding* if

$$\dim\left(T_p\,\sigma(N) \cap \mathcal{C}_p\right) \geqslant n-1 \qquad \text{for all} \quad p \in \sigma(N)\,.$$

Notation: $\sigma \in \mathcal{A}_N(\mathcal{E})$.

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ *defines a boundary value problem* if

$$T_p\,\sigma(\partial N) \subset \mathcal{C}_p \qquad \text{for all} \quad p \in \sigma(\partial N)\,.$$

Notation: $\sigma \in \mathcal{BA}_N(\mathcal{E})$.

Almost Cartan embeddings

Example

Consider the heat equation $u_t = u_{xx}$ and its infinite prolongation

$$\mathcal{E}\colon \qquad u_t - u_{xx} = 0, \quad D_x(u_t - u_{xx}) = 0, \quad D_t(u_t - u_{xx}) = 0, \quad \ldots$$

Suppose we have some initial conditions for all $t_0 \in \mathbb{R}$: $u = f(x, t_0)$. All the derivatives can be defined using these data:

$$u = f(x, t_0), \quad u_x = \partial_x f(x, t_0), \quad u_t = \partial_x^2 f(x, t_0), \quad u_{xx} = \partial_x^2 f(x, t_0), \ \ldots$$

Substituting t for t_0 in these formulas, we obtain an embedding of $\mathbb{R}^2$ to $\mathcal{E}$. The restriction of this embedding to a compact submanifold of $\mathbb{R}^2$ is an almost Cartan embedding.

Intrinsic integral functionals

Let $[L]_h$ be a horizontal n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ defines a unique integral functional

$$S\colon \mathcal{BA}_N(\mathcal{E}) \to \mathbb{R}\,, \qquad S(\sigma) = \int_N \sigma^*(l)\,,$$

where l represents the corresponding element of quotient (5). If σ defines a solution to $\mathcal{E}$, then $S(\sigma)$ coincide with the value of the original action on σ.

We can consider variations of S. *This explains why the intrinsic geometry of a variational system of equations knows about its variational nature.*

However, further, we define stationary points of internal Lagrangians (not of such integral functionals).

Intrinsic integral functionals

Let $[L]_h$ be a horizontal n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ defines a unique integral functional

$$S\colon \mathcal{BA}_N(\mathcal{E}) \to \mathbb{R}\,, \qquad S(\sigma) = \int_N \sigma^*(l)\,,$$

where l represents the corresponding element of quotient (5). If σ defines a solution to $\mathcal{E}$, then $S(\sigma)$ coincide with the value of the original action on σ.

We can consider variations of S. *This explains why the intrinsic geometry of a variational system of equations knows about its variational nature.*

However, further, we define stationary points of internal Lagrangians (not of such integral functionals).

Intrinsic integral functionals

Let $[L]_h$ be a horizontal n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ defines a unique integral functional

$$S\colon \mathcal{BA}_N(\mathcal{E}) \to \mathbb{R}\,, \qquad S(\sigma) = \int_N \sigma^*(l)\,,$$

where l represents the corresponding element of quotient (5). If σ defines a solution to $\mathcal{E}$, then $S(\sigma)$ coincide with the value of the original action on σ.

We can consider variations of S. *This explains why the intrinsic geometry of a variational system of equations knows about its variational nature.*

However, further, we define stationary points of internal Lagrangians (not of such integral functionals).

Intrinsic integral functionals

Let $[L]_h$ be a horizontal n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ defines a unique integral functional

$$S\colon \mathcal{BA}_N(\mathcal{E}) \to \mathbb{R}\,, \qquad S(\sigma) = \int_N \sigma^*(l)\,,$$

where l represents the corresponding element of quotient (5). If σ defines a solution to $\mathcal{E}$, then $S(\sigma)$ coincide with the value of the original action on σ.

We can consider variations of S. *This explains why the intrinsic geometry of a variational system of equations knows about its variational nature.*

However, further, we define stationary points of internal Lagrangians (not of such integral functionals).

Variation of an internal Lagrangian

Consider a path in $\mathcal{A}_N(\mathcal{E})$, that is, a smooth mapping $\gamma\colon \mathbb{R}\times N \to \mathcal{E}$ such that for all $\tau\in\mathbb{R}$ the mappings

$$\gamma(\tau)\colon N \to \mathcal{E}\,, \qquad \gamma(\tau)\colon x \mapsto \gamma(\tau, x) \tag{7}$$

are almost Cartan embeddings. Let 0_N denote the zero-section

$$0_N\colon N \to \mathbb{R}\times N, \qquad 0_N(x) = (0, x). \tag{8}$$

If the boundary is fixed, we obtain

$$\frac{d}{d\tau}\Big|_{\tau=0}\int_N \gamma(\tau)^*(l) = \int_N 0_N^*\big(i_{\partial_\tau}\gamma^*(dl)\big). \tag{9}$$

So, the derivative along a path γ is completely determined by the corresponding presymplectic structure $dl + \mathcal{C}^3\Lambda^{n+1}(\mathcal{E}) + d(\mathcal{C}^2\Lambda^{n+1}(\mathcal{E}))$.

Variation of an internal Lagrangian

Consider a path in $\mathcal{A}_N(\mathcal{E})$, that is, a smooth mapping $\gamma\colon \mathbb{R}\times N \to \mathcal{E}$ such that for all $\tau\in\mathbb{R}$ the mappings

$$\gamma(\tau)\colon N \to \mathcal{E}\,, \qquad \gamma(\tau)\colon x \mapsto \gamma(\tau, x) \tag{7}$$

are almost Cartan embeddings. Let 0_N denote the zero-section

$$0_N\colon N \to \mathbb{R}\times N, \qquad 0_N(x) = (0, x). \tag{8}$$

If the boundary is fixed, we obtain

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = \int_N 0_N^*\big(i_{\partial_\tau}\gamma^*(dl)\big). \tag{9}$$

So, the derivative along a path γ is completely determined by the corresponding presymplectic structure $dl + \mathcal{C}^3\Lambda^{n+1}(\mathcal{E}) + d(\mathcal{C}^2\Lambda^{n+1}(\mathcal{E}))$.

Diagram

$$\begin{tikzcd}
{[L]_h \ \ \text{such that} \ \ \mathrm{E}[L]_h|_{\mathcal{E}} = 0} \arrow[r] \arrow[d, shift right] & {[L]_h + d_h\Lambda_h^{n-1}(\pi)} \arrow[d, shift right] \\
\dfrac{\{l \in \Lambda^n(\mathcal{E})\colon \ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))} \arrow[u, dashed, bend right] \arrow[d] & \dfrac{\{l \in \Lambda^n(\mathcal{E})\colon \ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))} \arrow[u, dashed, bend right] \arrow[d, dashed] \\
S \arrow[d] & \text{ambiguous functional} \arrow[d] \\
\begin{array}{c}\text{unambiguous variations}\\ \text{along paths}\end{array} & \begin{array}{c}\text{unambiguous variations}\\ \text{along paths}\end{array}
\end{tikzcd}$$

Stationary points of an internal Lagrangian

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a *stationary point* of $\ell = l + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))$ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0 \tag{10}$$

for any path γ in $\mathcal{A}_N(\mathcal{E})$ such that $\gamma(0) = \sigma$ and the boundary is fixed.

If σ defines a solution to $\mathcal{E}$, then it is a stationary point of any internal Lagrangian of $\mathcal{E}$.

Apparently, one can also define stationary points of conservation laws the same way.

Stationary points of an internal Lagrangian

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a *stationary point* of $\boldsymbol{\ell} = l + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))$ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0 \tag{10}$$

for any path γ in $\mathcal{A}_N(\mathcal{E})$ such that $\gamma(0) = \sigma$ and the boundary is fixed.

If σ defines a solution to $\mathcal{E}$, then it is a stationary point of any internal Lagrangian of $\mathcal{E}$.

Apparently, one can also define stationary points of conservation laws the same way.

Stationary points of an internal Lagrangian

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a *stationary point* of $\boldsymbol{\ell} = l + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))$ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0 \tag{10}$$

for any path γ in $\mathcal{A}_N(\mathcal{E})$ such that $\gamma(0) = \sigma$ and the boundary is fixed.

If σ defines a solution to $\mathcal{E}$, then it is a stationary point of any internal Lagrangian of $\mathcal{E}$.

Apparently, one can also define stationary points of conservation laws the same way.

Example

Consider Laplace's equation $u_{xx} + u_{yy} = 0$ and its infinite prolongation $\mathcal{E}$. $N \subset \mathbb{R}^2$ is a compact submanifold.

$$L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy\,. \tag{11}$$

The corresponding internal Lagrangian is represented by the restriction of

$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{12}$$

Suppose $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a (local) section of the bundle $\pi_\infty|_{\mathcal{E}}$ such that

$$T_p\, \sigma(N) \ni \bar{D}_x|_p \qquad \text{for all} \quad p \in \sigma(N). \tag{13}$$

Then, in local coordinates on $\mathcal{E}$

$$\sigma: \quad u = f\,, \quad u_x = \partial_x f\,, \quad u_y = g\,, \quad u_{xx} = \partial_x^2 f\,, \quad u_{xy} = \partial_x g\,, \quad \ldots \tag{14}$$

Arbitrary smooth functions $f, g \colon N \to \mathbb{R}$ determine an appropriate σ and vice versa.

Example

Consider Laplace's equation $u_{xx} + u_{yy} = 0$ and its infinite prolongation $\mathcal{E}$. $N \subset \mathbb{R}^2$ is a compact submanifold.

$$L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy\,. \tag{11}$$

The corresponding internal Lagrangian is represented by the restriction of

$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{12}$$

Suppose $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a (local) section of the bundle $\pi_\infty|_{\mathcal{E}}$ such that

$$T_p\, \sigma(N) \ \ni\ \bar{D}_x|_p \qquad \text{for all} \quad p \in \sigma(N). \tag{13}$$

Then, in local coordinates on $\mathcal{E}$

$$\sigma: \quad u = f\,, \quad u_x = \partial_x f\,, \quad u_y = g\,, \quad u_{xx} = \partial_x^2 f\,, \quad u_{xy} = \partial_x g\,, \quad \dots \tag{14}$$

Arbitrary smooth functions $f, g\colon N \to \mathbb{R}$ determine an appropriate σ and vice versa.

Example

Consider the path in local sections satisfying condition (13)

$$\gamma\colon \qquad u = f + \tau\delta f\,, \qquad u_x = \partial_x(f + \tau\delta f)\,, \qquad u_y = g + \tau\delta g\,, \qquad \ldots$$

Here δf and δg denote arbitrary smooth functions on N vanishing together with all their derivatives on ∂N. It suffices to consider the pullback

$$\gamma(0)^*(l) = \Big(-\frac{(\partial_x f)^2 - g^2}{2} - g\,\partial_y f\Big)\,dx \wedge dy\,, \tag{15}$$

where $l = (L + \omega_L)|_{\mathcal{E}}$. The corresponding Euler-Lagrange equations are

$$\partial_x^2 f + \partial_y g = 0\,, \qquad g - \partial_y f = 0\,. \tag{16}$$

Therefore, an almost Cartan embedding $\sigma \in \mathcal{A}_N(\mathcal{E})$ that is a local section of the bundle $\pi_\infty|_{\mathcal{E}}$ and satisfies (13) is a stationary point of the internal Lagrangian under consideration iff it defines a solution to $\mathcal{E}$.

Furthermore, in order to obtain this result, it suffices to consider only paths that satisfy condition (13) for all $\tau \in \mathbb{R}$ and pass through local sections. As we will see below, this is not a coincidence.

Example

Consider the path in local sections satisfying condition (13)

$$\gamma: \qquad u = f + \tau\delta f\,, \qquad u_x = \partial_x(f + \tau\delta f)\,, \qquad u_y = g + \tau\delta g\,, \qquad \ldots$$

Here δf and δg denote arbitrary smooth functions on N vanishing together with all their derivatives on ∂N. It suffices to consider the pullback

$$\gamma(0)^*(l) = \Big(-\frac{(\partial_x f)^2 - g^2}{2} - g\,\partial_y f \Big)\, dx \wedge dy\,, \tag{15}$$

where $l = (L + \omega_L)|_{\mathcal{E}}$. The corresponding Euler-Lagrange equations are

$$\partial_x^2 f + \partial_y g = 0\,, \qquad g - \partial_y f = 0\,. \tag{16}$$

Therefore, an almost Cartan embedding $\sigma \in \mathcal{A}_N(\mathcal{E})$ that is a local section of the bundle $\pi_\infty|_{\mathcal{E}}$ and satisfies (13) is a stationary point of the internal Lagrangian under consideration iff it defines a solution to $\mathcal{E}$.

Furthermore, in order to obtain this result, it suffices to consider only paths that satisfy condition (13) for all $\tau \in \mathbb{R}$ and pass through local sections. As we will see below, this is not a coincidence.

Example

Consider the path in local sections satisfying condition (13)

$$\gamma\colon \qquad u = f + \tau\delta f\,, \qquad u_x = \partial_x(f + \tau\delta f)\,, \qquad u_y = g + \tau\delta g\,, \qquad \ldots$$

Here δf and δg denote arbitrary smooth functions on N vanishing together with all their derivatives on ∂N. It suffices to consider the pullback

$$\gamma(0)^*(l) = \Big(-\frac{(\partial_x f)^2 - g^2}{2} - g\,\partial_y f\Big)\,dx \wedge dy\,, \tag{15}$$

where $l = (L + \omega_L)|_{\mathcal{E}}$. The corresponding Euler-Lagrange equations are

$$\partial_x^2 f + \partial_y g = 0\,, \qquad g - \partial_y f = 0\,. \tag{16}$$

Therefore, an almost Cartan embedding $\sigma \in \mathcal{A}_N(\mathcal{E})$ that is a local section of the bundle $\pi_\infty|_{\mathcal{E}}$ and satisfies (13) is a stationary point of the internal Lagrangian under consideration iff it defines a solution to $\mathcal{E}$.

Furthermore, in order to obtain this result, it suffices to consider only paths that satisfy condition (13) for all $\tau \in \mathbb{R}$ and pass through local sections. As we will see below, this is not a coincidence.

Example

Consider the path in local sections satisfying condition (13)

$$\gamma: \qquad u = f + \tau\delta f\,, \qquad u_x = \partial_x(f + \tau\delta f)\,, \qquad u_y = g + \tau\delta g\,, \qquad \ldots$$

Here δf and δg denote arbitrary smooth functions on N vanishing together with all their derivatives on ∂N. It suffices to consider the pullback

$$\gamma(0)^*(l) = \Big(-\frac{(\partial_x f)^2 - g^2}{2} - g\,\partial_y f\Big)\,dx \wedge dy\,, \tag{15}$$

where $l = (L + \omega_L)|_{\mathcal{E}}$. The corresponding Euler-Lagrange equations are

$$\partial_x^2 f + \partial_y g = 0\,, \qquad g - \partial_y f = 0\,. \tag{16}$$

Therefore, an almost Cartan embedding $\sigma \in \mathcal{A}_N(\mathcal{E})$ that is a local section of the bundle $\pi_\infty|_{\mathcal{E}}$ and satisfies (13) is a stationary point of the internal Lagrangian under consideration iff it defines a solution to $\mathcal{E}$.

Furthermore, in order to obtain this result, it suffices to consider only paths that satisfy condition (13) for all $\tau \in \mathbb{R}$ and pass through local sections. As we will see below, this is not a coincidence.

Noether's theorem

Infinitesimal symmetries of an infinitely prolonged system of equations $\mathcal{E}$ act on its internal Lagrangians by means of the Lie derivative.

Internal Lagrangians are related to presymplectic structures, that is, elements of the kernel of the differential

$$d_1^{2,\,n-1} \colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E})\,.$$

Here $E_r^{p,\,q}(\mathcal{E})$ are groups of the Vinogradov $\mathcal{C}$-spectral sequence.

Theorem

Let ℓ be an internal Lagrangian, and let X be an infinitesimal symmetry of an infinitely prolonged system of differential equations $\mathcal{E}$. If ℓ is invariant under the action of X, then X gives rise to conservation laws. Otherwise, X produces the non-trivial internal Lagrangian $\mathcal{L}_X\,\ell$.

Besides, if $\mathcal{E}$ admits gauge symmetries, then all its internal Lagrangians are gauge invariant.

Noether's theorem

Infinitesimal symmetries of an infinitely prolonged system of equations $\mathcal{E}$ act on its internal Lagrangians by means of the Lie derivative.

Internal Lagrangians are related to presymplectic structures, that is, elements of the kernel of the differential

$$d_1^{2,\,n-1}\colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E})\,.$$

Here $E_r^{p,\,q}(\mathcal{E})$ are groups of the Vinogradov $\mathcal{C}$-spectral sequence.

Theorem

Let ℓ be an internal Lagrangian, and let X be an infinitesimal symmetry of an infinitely prolonged system of differential equations $\mathcal{E}$. If ℓ is invariant under the action of X, then X gives rise to conservation laws. Otherwise, X produces the non-trivial internal Lagrangian $\mathcal{L}_X\,\ell$.

Besides, if $\mathcal{E}$ admits gauge symmetries, then all its internal Lagrangians are gauge invariant.

Noether's theorem

Infinitesimal symmetries of an infinitely prolonged system of equations $\mathcal{E}$ act on its internal Lagrangians by means of the Lie derivative.

Internal Lagrangians are related to presymplectic structures, that is, elements of the kernel of the differential

$$d_1^{2,\,n-1}\colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E})\,.$$

Here $E_r^{p,\,q}(\mathcal{E})$ are groups of the Vinogradov $\mathcal{C}$-spectral sequence.

Theorem

Let ℓ be an internal Lagrangian, and let X be an infinitesimal symmetry of an infinitely prolonged system of differential equations $\mathcal{E}$. If ℓ is invariant under the action of X, then X gives rise to conservation laws. Otherwise, X produces the non-trivial internal Lagrangian $\mathcal{L}_X\,\ell$.

Besides, if $\mathcal{E}$ admits gauge symmetries, then all its internal Lagrangians are gauge invariant.

ξ-stationary points

Suppose $N \subset M^n$ is a compact oriented n-dimensional submanifold, where M is a base of some jets bundle π_∞. By $\pi_\mathcal{E}$ we denote the restriction $\pi_\infty|_\mathcal{E}$.

Definition

We say that $\sigma \in \mathcal{A}_N(\mathcal{E})$ is an *almost Cartan section* of $\pi_\mathcal{E}$ if $\pi_\mathcal{E} \circ \sigma = \mathrm{id}_N$.

Definition

Let $\xi \in \Lambda^1(M)$ be a covector field. An almost Cartan section $\sigma \colon N \to \mathcal{E}$ is a *ξ-section* of $\pi_\mathcal{E}$ if $d\sigma(\ker \xi|_x) \subset \mathcal{C}_{\sigma(x)}$ for all $x \in N$.

Definition

Let ℓ be an internal Lagrangian of $\mathcal{E}$, $l \in \ell$ be a differential form representing ℓ. We say that a ξ-section σ is a *ξ-stationary point* of ℓ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0$$

for any path in ξ-sections such that $\gamma(0) = \sigma$ and the boundary is fixed.

ξ-stationary points

Suppose $N \subset M^n$ is a compact oriented n-dimensional submanifold, where M is a base of some jets bundle π_∞. By $\pi_\mathcal{E}$ we denote the restriction $\pi_\infty|_\mathcal{E}$.

Definition

We say that $\sigma \in \mathcal{A}_N(\mathcal{E})$ is an *almost Cartan section* of $\pi_\mathcal{E}$ if $\pi_\mathcal{E} \circ \sigma = \mathrm{id}_N$.

Definition

Let $\xi \in \Lambda^1(M)$ be a covector field. An almost Cartan section $\sigma \colon N \to \mathcal{E}$ is a ξ-*section* of $\pi_\mathcal{E}$ if $d\sigma(\ker \xi|_x) \subset \mathcal{C}_{\sigma(x)}$ for all $x \in N$.

Definition

Let ℓ be an internal Lagrangian of $\mathcal{E}$, $l \in \ell$ be a differential form representing ℓ. We say that a ξ-section σ is a ξ-*stationary point* of ℓ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0$$

for any path in ξ-sections such that $\gamma(0) = \sigma$ and the boundary is fixed.

Non-degenerate Lagrangians

If a ξ-section is a stationary point of an internal Lagrangian, it is also a ξ-stationary one.

Theorem

Let L be a differential n-form on $J^k(\pi)$, $k \geqslant 1$, and let $\mathcal{E}$ be the infinite prolongation of the Euler-Lagrange equations $\mathrm{E}[L]_h = 0$. Suppose $\xi \in \Lambda^1(M)$ is a non-vanishing, non-characteristic covector field such that the distribution $\xi = 0$ is integrable. Then a ξ-section σ is a ξ-stationary point of the corresponding internal Lagrangian if and only if σ is a (local) solution to $\pi_{\mathcal{E}}$.

Unfortunately, this theorem is inapplicable to gauge theories since they admit only characteristic covector fields. However, internal Lagrangians can be restricted to subsystems that arise after gauge fixing. Perhaps the approach still needs some modification in this case.

Besides, the Proca theory and the massive spin-2 theory are examples of degenerate non-gauge Euler-Lagrange equations.

Non-degenerate Lagrangians

If a ξ-section is a stationary point of an internal Lagrangian, it is also a ξ-stationary one.

Theorem

Let L be a differential n-form on $J^k(\pi)$, $k \geqslant 1$, and let $\mathcal{E}$ be the infinite prolongation of the Euler-Lagrange equations $\mathrm{E}[L]_h = 0$. Suppose $\xi \in \Lambda^1(M)$ is a non-vanishing, non-characteristic covector field such that the distribution $\xi = 0$ is integrable. Then a ξ-section σ is a ξ-stationary point of the corresponding internal Lagrangian if and only if σ is a (local) solution to $\pi_{\mathcal{E}}$.

Unfortunately, this theorem is inapplicable to gauge theories since they admit only characteristic covector fields. However, internal Lagrangians can be restricted to subsystems that arise after gauge fixing. Perhaps the approach still needs some modification in this case.

Besides, the Proca theory and the massive spin-2 theory are examples of degenerate non-gauge Euler-Lagrange equations.

Non-degenerate Lagrangians

If a ξ-section is a stationary point of an internal Lagrangian, it is also a ξ-stationary one.

Theorem

Let L be a differential n-form on $J^k(\pi)$, $k \geqslant 1$, and let $\mathcal{E}$ be the infinite prolongation of the Euler-Lagrange equations $\mathrm{E}[L]_h = 0$. Suppose $\xi \in \Lambda^1(M)$ is a non-vanishing, non-characteristic covector field such that the distribution $\xi = 0$ is integrable. Then a ξ-section σ is a ξ-stationary point of the corresponding internal Lagrangian if and only if σ is a (local) solution to $\pi_{\mathcal{E}}$.

Unfortunately, this theorem is inapplicable to gauge theories since they admit only characteristic covector fields. However, internal Lagrangians can be restricted to subsystems that arise after gauge fixing. Perhaps the approach still needs some modification in this case.

Besides, the Proca theory and the massive spin-2 theory are examples of degenerate non-gauge Euler-Lagrange equations.

A surprising example

Consider the potential KdV equation

$$u_t = 3u_x^2 + u_{xxx}$$

and its infinite prolongation $\mathcal{E}$. This equation admits the presymplectic operator $\Delta = \bar{D}_x$ with non-trivial kernel. The corresponding presymplectic structure should be considered a degenerate one. It can be related only to a variational principle that gives a consequence of the original equation, but not the potential KdV itself.

There exists a unique internal Lagrangian ℓ producing the same presymplectic structure:

$$l = \Big(\frac{u_x u_t}{2} - u_x^3 + \frac{u_{xx}^2}{2}\Big) dt \wedge dx - \frac{1}{2} u_t \, dt \wedge \theta_0 + u_{xx} \, dt \wedge \theta_x + \frac{1}{2} u_x \, \theta_0 \wedge dx \,,$$

where $\theta_0 = du - u_x \, dx - u_t \, dt$ and $\theta_x = du_x - u_{xx} \, dx - u_{xt} \, dt$. Here we regard the variable u_{xxx} and its derivatives as external coordinates for $\mathcal{E}$.

A surprising example

Consider the potential KdV equation

$$u_t = 3u_x^2 + u_{xxx}$$

and its infinite prolongation $\mathcal{E}$. This equation admits the presymplectic operator $\Delta = \bar{D}_x$ with non-trivial kernel. The corresponding presymplectic structure should be considered a degenerate one. It can be related only to a variational principle that gives a consequence of the original equation, but not the potential KdV itself.

There exists a unique internal Lagrangian ℓ producing the same presymplectic structure:

$$l = \Big(\frac{u_x u_t}{2} - u_x^3 + \frac{u_{xx}^2}{2}\Big) dt \wedge dx - \frac{1}{2} u_t \, dt \wedge \theta_0 + u_{xx} \, dt \wedge \theta_x + \frac{1}{2} u_x \, \theta_0 \wedge dx \,,$$

where $\theta_0 = du - u_x \, dx - u_t \, dt$ and $\theta_x = du_x - u_{xx} \, dx - u_{xt} \, dt$. Here we regard the variable u_{xxx} and its derivatives as external coordinates for $\mathcal{E}$.

Suppose $N \subset \mathbb{R}^2$ is a connected compact 2-dimensional submanifold. Let us define ξ by

$$\xi = dx - X(t,x)dt\,. \tag{17}$$

Then a ξ-section σ is of the form

$$\sigma: \qquad \begin{array}{lll} u = f, & u_x = g, & u_t = \partial_t f + X(\partial_x f - g), \\ u_{xx} = h, & u_{xt} = \partial_t g + X(\partial_x g - h), & \dots \end{array} \tag{18}$$

Here f, g and h are arbitrary functions on N. The expressions for all other coordinates on $\mathcal{E}$ are unambiguously defined.

So, we get the pullback

$$\sigma^*(l) = \Big(-\frac{\partial_x f\,\partial_t f}{2} + g\,\partial_t f - g^3 + h\,\partial_x g - \frac{h^2}{2} - \frac{X}{2}(\partial_x f - g)^2 \Big) dt \wedge dx\,.$$

Varying the corresponding action with respect to f, g and h, we obtain

$$\begin{aligned} &\partial_t(\partial_x f - g) + \partial_x\big(X(\partial_x f - g)\big) = 0\,,\\ &\partial_t f - 3g^2 - \partial_x h + X(\partial_x f - g) = 0\,,\\ &\partial_x g - h = 0\,. \end{aligned} \tag{19}$$

These equations do not imply the relation $\partial_x f = g$. Therefore, the set of ξ-stationary points contains more than just local solutions. However, we can also vary with respect to X. As a result, we get the missing equation

$$\partial_x f = g\,. \tag{20}$$

Thus, if ξ determines a non-characteristic distribution, then a ξ-section σ is a stationary point of the internal Lagrangian ℓ iff σ is a local solution. But this is not the case for ξ-stationary points.

So, we get the pullback

$$\sigma^*(l) = \Big(-\frac{\partial_x f\, \partial_t f}{2} + g\, \partial_t f - g^3 + h\, \partial_x g - \frac{h^2}{2} - \frac{X}{2}(\partial_x f - g)^2 \Big) dt \wedge dx\,.$$

Varying the corresponding action with respect to f, g and h, we obtain

$$\begin{aligned} &\partial_t(\partial_x f - g) + \partial_x\big(X(\partial_x f - g)\big) = 0\,, \\ &\partial_t f - 3g^2 - \partial_x h + X(\partial_x f - g) = 0\,, \\ &\partial_x g - h = 0\,. \end{aligned} \tag{19}$$

These equations do not imply the relation $\partial_x f = g$. Therefore, the set of ξ-stationary points contains more than just local solutions. However, we can also vary with respect to X. As a result, we get the missing equation

$$\partial_x f = g\,. \tag{20}$$

Thus, if ξ determines a non-characteristic distribution, then a ξ-section σ is a stationary point of the internal Lagrangian ℓ iff σ is a local solution. But this is not the case for ξ-stationary points.

Literature

- K. Druzhkov, Lagrangian formalism and the intrinsic geometry of PDEs, J. Geom. Phys. 189 (2023) 104848.
- A.M. Vinogradov, I.S. Krasil'schik (eds.), Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Vol. 182, American Mathematical Society, 1999.
- P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer-Verlag, 1993.
- A.M. Vinogradov, The $\mathcal{C}$-spectral sequence, Lagrangian formalism and conservation laws: I the linear theory; II the non-linear theory, J. Math. Anal. and Appl. 100 (1984) 1–40, 41–129.
- M. Grigoriev, Presymplectic structures and intrinsic Lagrangians, arXiv:1606.07532 (2016).

Literature

- M. Grigoriev, V. Gritzaenko, Presymplectic structures and intrinsic Lagrangians for massive fields, Nucl. Phys. B (2022) 975(4):115686.
- J. Krasil'shchik, A.M. Verbovetsky, Homological methods in equations of mathematical physics, in: Advanced Texts in Mathematics, Open Education & Sciences, Opava, 1998. arXiv:math/9808130.
- M. Fierz, W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. Roy. Soc. Lond. A173 (1939) 211–232.
- I. Dorfman, Dirac Structures and Integrability of Nonlinear Evolution Equations, Vol. 176, JOHN WILEY & SONS, 1993.

Thanks a lot for your attention!