

Recent Developments in gauge/YBE Correspondence

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$$\mathcal{Z}_{S_b^3/\mathbb{Z}_r} \begin{array}{l} \text{Gauge} = SU(2) \\ \text{Flavour Group} : SU(6) \end{array} = \mathcal{Z}_{S_b^3/\mathbb{Z}_r} \begin{array}{l} \text{No Gauge Symmetry} \\ 15 \text{ Chiral Multiplets} \end{array}$$

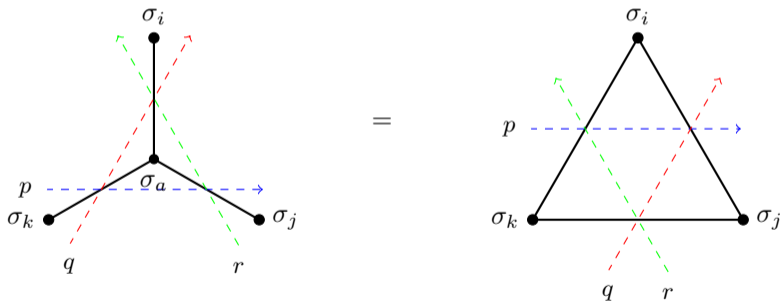
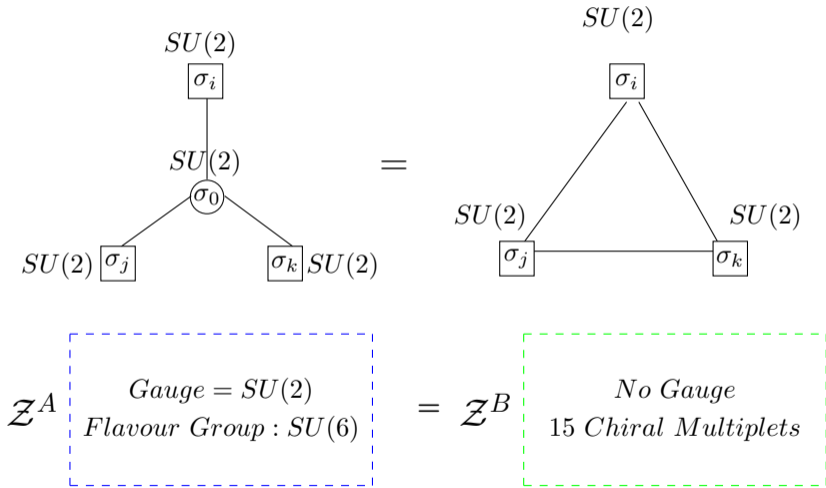


Figure: See [Yamazaki, 2018] and [Gahramanov and Shahriyar, 2017] for comprehensive review



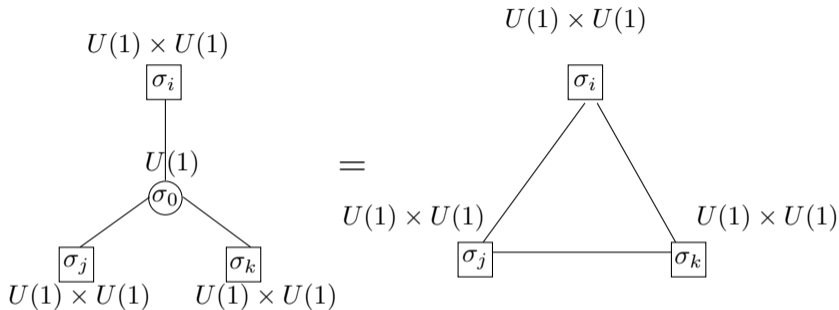
$$\sum_{m=0}^{\lfloor r/2 \rfloor} \epsilon(m) \int_{-\infty}^{\infty} \frac{\prod_{i=1}^6 \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2)}{\gamma_h(\pm 2ix, \pm 2m; \omega_1, \omega_2)} \frac{dx}{2r\sqrt{-\omega_1\omega_2}} = \prod_{1 \leq i < j \leq 6} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)$$

where balancing conditions are $\sum_{i=1}^6 a_i = \omega_1 + \omega_2$ and $\sum_{i=1}^6 u_i = 0$ and the Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2) \quad (1)$$

The Boltzmann weight satisfies the following star-triangle relation

$$\begin{aligned} \sum_{\sigma_0} \int dx_0 \bar{W}_{qr}(\sigma_i, \sigma_0) W_{pr}(\sigma_j, \sigma_0) \bar{W}_{pq}(\sigma_k, \sigma_0) \\ = R(p, q, r) W_{pq}(\sigma_j, \sigma_k) \bar{W}_{pr}(\sigma_i, \sigma_k) W_{qr}(\sigma_j, \sigma_i). \end{aligned} \quad (2)$$



Obtained by gauge symmetry breaking [Spiridonov, 2010]

$$\mathcal{Z} \left[\begin{array}{l} \text{Gauge} = U(1) \\ 6 \text{ Chiral Multiplets} \\ \text{Global Symmetry :} \\ SU(3) \times SU(3) \times U(1) \end{array} \right] = \mathcal{Z} \left[\begin{array}{l} \text{No Gauge} \\ 9 \text{ Free Mesons} \\ \text{Global Symmetry :} \\ SU(3) \times SU(3) \times U(1) \end{array} \right]$$

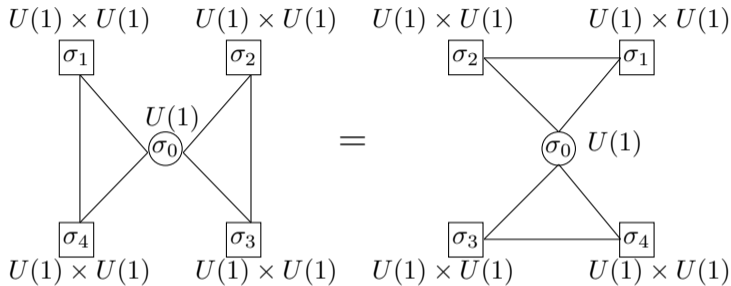
$$\sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \prod_{i=1}^3 \gamma_h(a_i - x, u_i - m; \omega_1, \omega_2) \gamma_h(b_i + x, v_i + m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}}$$

$$= \prod_{i,j=1}^3 \gamma_h(a_i + b_j, u_i + v_j; \omega_1, \omega_2) \quad (3)$$

where balancing conditions are $\sum_{i=1}^3 a_i + b_i = \omega_1 + \omega_2$ and $\sum_{i=1}^3 u_i + v_i = 0$ and the Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = e^{-\pi i(u_i + u_j)} \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2)$$

$$\gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2). \quad (4)$$



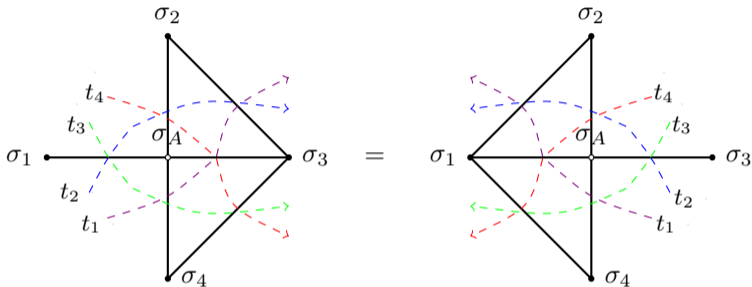
$$\mathcal{Z} \quad \begin{array}{l} \text{Gauge} = U(1) \\ \text{Flavour Symmetry :} \\ SU(4) \times SU(4) \times U(1) \end{array}$$

$$= \mathcal{Z} \quad \begin{array}{l} \text{Gauge} = U(1) \\ \text{Global Symmetry :} \\ [SU(2) \times SU(2)]^2 \times U(1) \end{array}$$

$$\begin{aligned}
 & \sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \prod_{i=1}^8 \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}} \\
 = & \prod_{1 \leq i < j \leq 4} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2) \gamma_h(a_{i+4} + a_{j+4}, u_{i+4} + u_{j+4}; \omega_1, \omega_2) \\
 & \times \sum_{y=0}^{[r/2]} \epsilon(y) \int_{-\infty}^{\infty} \prod_{i=1}^8 \gamma_h(\tilde{a}_i \pm z, \tilde{u}_i \pm y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}} \quad (5)
 \end{aligned}$$

where balancing conditions are $\sum_{i=1}^8 a_i = 2(\omega_1 + \omega_2)$ and $\sum_{i=1}^8 u_i = 0$ and Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2). \quad (6)$$



$$R_{(t_{34}t_{21})} \begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} = R_{(t_{21}t_{34})} \begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} \quad (7)$$

$$\begin{aligned}
 \sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \prod_{i=1}^4 \gamma_h(a_i - x, u_i - m; \omega_1, \omega_2) \gamma_h(b_i + x, v_i + m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}} \\
 = e^{-\pi i \sum_{i=1}^2 (u_i + u_j)} \prod_{i,j=1}^2 \gamma_h(a_i + b_j, u_i + u_j; \omega_1, \omega_2) \\
 \sum_{y=0}^{[r/2]} \epsilon(y) \int_{-\infty}^{\infty} \prod_{i=1}^4 \gamma_h(\tilde{a}_i - z, \tilde{u}_i - y; \omega_1, \omega_2) \gamma_h(\tilde{b}_i + z, \tilde{v}_i + y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}} \quad (8)
 \end{aligned}$$

where balancing conditions are $\sum_{i=1}^4 a_i + b_i = 2(\omega_1 + \omega_2)$ and $\sum_{i=1}^4 u_i + v_i = 0$ and Boltzmann weights are

$$\begin{aligned}
 W_{pq}(\sigma_i, \sigma_j) = e^{-\pi i (u_i + u_j)} \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2) \\
 \gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2). \quad (9)
 \end{aligned}$$

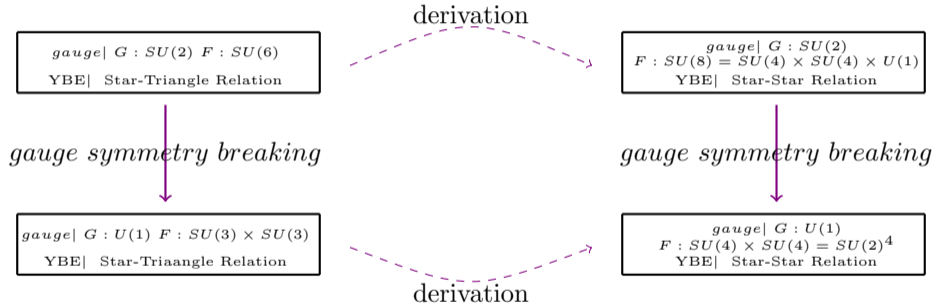
What is the meaning of the gauge symmetry breaking from the statistical mechanics point of view?

The Boltzmann weight for $SU(2)$ gauge symmetry

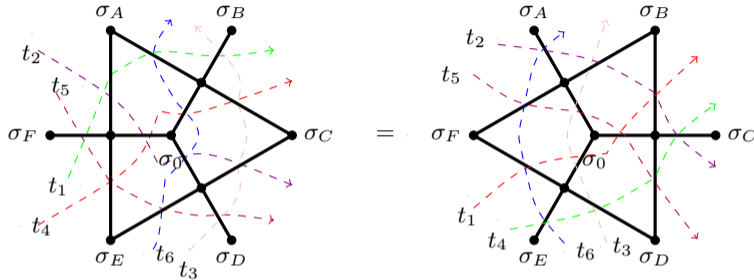
$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2) \quad (10)$$

The Boltzmann weight for $U(1)$ gauge symmetry

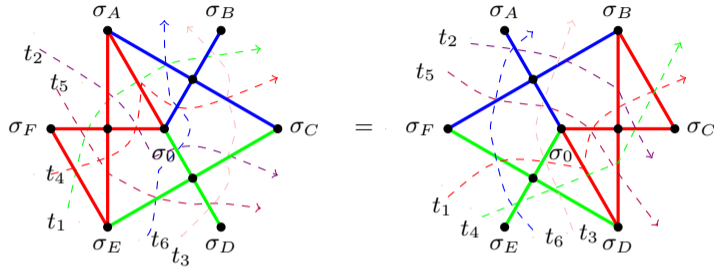
$$W_{pq}(\sigma_i, \sigma_j) = e^{-\pi i(u_i + u_j)} \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2) \gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2). \quad (11)$$



Could we obtain IRF-type YBE
from the edge interacting models?



$$\begin{aligned}
 & \sum_{m_0} \int dx_0 R_{t_{41}t_{63}} \begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_0 & \sigma_C \end{pmatrix} R_{t_{63}t_{25}} \begin{pmatrix} \sigma_C & \sigma_D \\ \sigma_0 & \sigma_E \end{pmatrix} R_{t_{25}t_{41}} \begin{pmatrix} \sigma_E & \sigma_F \\ \sigma_0 & \sigma_A \end{pmatrix} \\
 & = \sum_{m_0} \int dx_0 R_{t_{63}t_{25}} \begin{pmatrix} \sigma_B & \sigma_0 \\ \sigma_A & \sigma_F \end{pmatrix} R_{t_{25}t_{41}} \begin{pmatrix} \sigma_D & \sigma_0 \\ \sigma_C & \sigma_B \end{pmatrix} R_{t_{41}t_{63}} \begin{pmatrix} \sigma_F & \sigma_0 \\ \sigma_E & \sigma_D \end{pmatrix}
 \end{aligned} \tag{12}$$



$$\begin{aligned}
 & \sum_{m_0 \in \mathbb{Z}} \int dx_0 \, R_{t_{25}t_{41}} \begin{pmatrix} & \sigma_A & \\ \sigma_F & & \sigma_0 \\ & \sigma_E & \end{pmatrix} R_{t_{63}t_{25}} \begin{pmatrix} \sigma_0 & \sigma_C \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{41}t_{63}} \begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_0 & \sigma_C \end{pmatrix} \\
 &= \sum_{m_0 \in \mathbb{Z}} \int dx_0 \, R_{t_{41}t_{63}} \begin{pmatrix} \sigma_F & \sigma_0 \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{63}t_{25}} \begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_F & \sigma_0 \end{pmatrix} R_{t_{25}t_{41}} \begin{pmatrix} & \sigma_B & \\ \sigma_0 & & \sigma_C \\ & \sigma_D & \end{pmatrix}
 \end{aligned}$$

If one has IRF-type models,
expects to acquire vertex-type models?

Lemma (Bailey Lemma)

Suppose $\alpha(x, m; t, p)$ and $\beta(x, m; t, p)$ form an integral Bailey pair with respect to $t \in \mathbb{C}$ and $p \in \mathbb{Z}$. Then, the sequences of functions $\alpha'(x, k; t + s, p + q)$ and $\beta'(x, k; t + s, p + q)$, $k \in \mathbb{Z}$, defined by

$$\alpha'(x, k; t + s, p + q) = D(s, q; y, l; x, k)\alpha(x, k; t, p), \quad (13)$$

$$\beta'(x, k; t + s, p + q) = D(-t, -p; y, l; x, k)M(s, q)_{x, k; z, m}D(s + t, p + q; y, l; z, m)\beta(z, m; t, p), \quad (14)$$

form a Bailey pair with respect to the new parameters $t + s$ and $p + q$ where $s, y \in \mathbb{C}$, $q, l \in \mathbb{Z}$ are arbitrary and the operator $D(s, q; y, l; x, k)$ is described as above.

$$\begin{aligned} M(s, q)_{w, k; z, m}D(s + t, q + p; y, l; z, m)M(t, p)_{z, m; x, j} \\ = D(t, p; y, l; w, k)M(s + t, q + p)_{w, k; x, j}D(s, q; y, l; x, j). \end{aligned} \quad (15)$$

$$M(t, p)_{z, m; x, j} = \frac{1}{C(t, p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2) \frac{[d_j x]}{2r \sqrt{-\omega_1 \omega_2}}, \quad (16)$$

$$D(t, p; x, j; z, m) = \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2)$$

The same operators satisfy the star-star relation.
How does gauge symmetry breaking work for the operators?

$$M(t, p)_{z, m; x, j} = \frac{1}{C(t, p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t + z + x, m - p + j; \omega_1, \omega_2) \\ \times \gamma_h(-t - z - x, -m - p - j; \omega_1, \omega_2) \frac{[d_j x]}{2r \sqrt{-\omega_1 \omega_2}}, \quad (17)$$

$$D(t, p; x, j; z, m) = \gamma_h(-t + z + x, m - p + j; \omega_1, \omega_2) \\ \times \gamma_h(-t - z - x, -m - p - j; \omega_1, \omega_2)$$

What about higher-spin interacting lattice models
from the supersymmetric gauge theories?

How do we interpret the followings?

$$\begin{aligned}
 & \sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \prod_{i=1}^8 \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}} \\
 & \qquad \qquad \qquad = \prod_{1 \leq i < j \leq 8} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2) \\
 & \times \sum_{y=0}^{[r/2]} \epsilon(y) \int_{-\infty}^{\infty} \prod_{i=1}^8 \gamma_h(\tilde{a}_i \pm z, \tilde{u}_i \pm y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}} \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{m=0}^{[r/2]} \epsilon(m) \int_{-\infty}^{\infty} \frac{\prod_{i=1}^4 \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2)}{\gamma_h(\pm 2ix, \pm 2m; \omega_1, \omega_2)} \frac{dx}{2r\sqrt{-\omega_1\omega_2}} \\
 & \qquad \qquad \qquad = \gamma_h \left(\sum_{i=1}^4 a_i, \sum_{i=1}^4 u_i; \omega_1, \omega_2 \right) \prod_{1 \leq i < j \leq 4} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)
 \end{aligned}$$

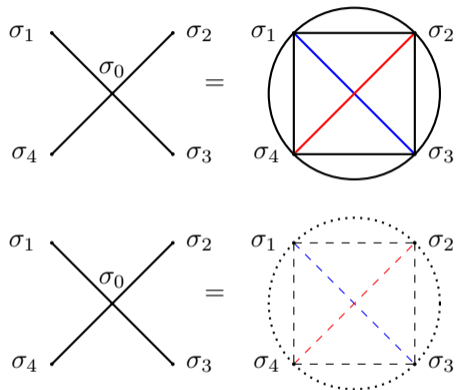


Figure: The star-square relation consists of four nearest neighbor interactions at left and seven interactions which are four nearest neighbors (dotted lines), two next nearest neighbors (dashed lines), and one quadruple interaction (broken circle) at right.

Why does the absence of gauge symmetry break
the Boltzmann weights of non-planar model?

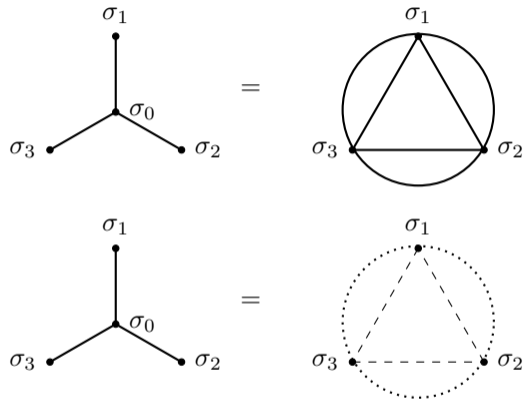


Figure: The generalized star-triangle relation consists of three nearest neighbor interactions at the left and four interactions which are three nearest neighbors (dashed lines) and one triple interaction (broken circle) at the right.

Introducing one dimensional relations extends
the gauge/YBE correspondence.



Figure: The decoration transformation

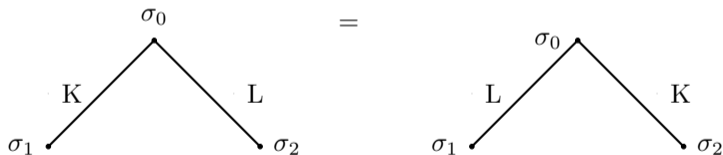


Figure: The flipping relation.

The decoration transformation for IRF-type models

$$\sum_{m_0} \int dx_0 R \begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} R \begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} = R \begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix} \quad (19)$$

The flipping relation for IRF-type models

$$\sum_{m_0} \int dx_0 R \begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} R \begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} = \sum_{m_0} \int dx_0 R \begin{pmatrix} \sigma & \sigma_0 \\ \sigma & \sigma \end{pmatrix} R \begin{pmatrix} \sigma & \sigma_0 \\ \sigma & \sigma \end{pmatrix} \quad (20)$$

The decoration transformation for vertex-type models

$$M(s, q)_{w, k; z, m} M(t, p)_{z, m; x, j} = M(s + t, q + p)_{w, k; x, j} \cdot \quad (21)$$

The flipping relation for vertex-type models

$$M(c, d)_{w, k; z, m} M(s + t, q + p)_{w, k; x, j} = M(s + c, q + d)_{w, k; x, j} M(t, p)_{w, k; x, j} \cdot \quad (22)$$

$$M(t, p)_{z, m; x, j} = \frac{1}{C(t, p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2) \frac{[d_j x]}{2r\sqrt{-\omega_1\omega_2}}, \quad (23)$$

$$D(t, p; x, j; z, m) = I(t, p)$$

The operators satisfy both the decoration transformation and the flipping relation.

The gauge symmetry breaking works again, WHY?

$$M(t, p)_{z, m; x, j} = \frac{1}{C(t, p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t + z + x, m - p + j; \omega_1, \omega_2) \\ \times \gamma_h(-t - z - x, -m - p - j; \omega_1, \omega_2) \frac{[d_j x]}{2r\sqrt{-\omega_1\omega_2}}, \quad (24)$$

$$D(t, p; x, j; z, m) = I(t, p)$$

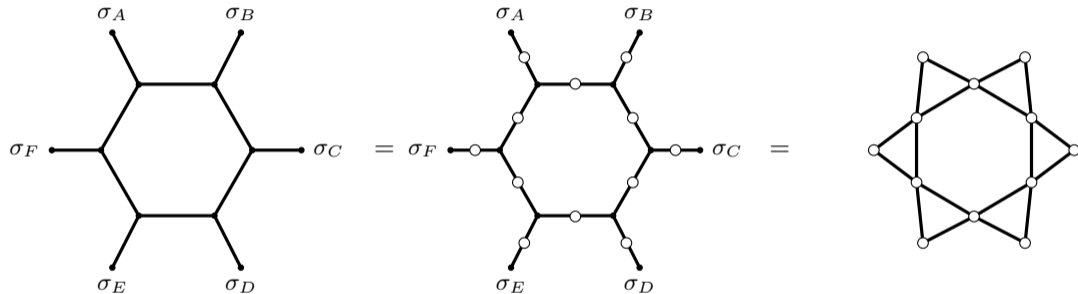


Figure: Constructing Kagome lattice from the hexagonal lattice by the use of decoration transformation and the star-triangle relation, respectively.

The infinite product representation is

$$\gamma^{(2)}(z; \omega_1, \omega_2) = e^{\frac{\pi i}{2} B_{2,2}(z; \omega_1, \omega_2)} \frac{(e^{2\pi i \frac{z}{\omega_2}} \tilde{q}; \tilde{q})_\infty}{(e^{2\pi i \frac{z}{\omega_1}} q)_\infty} = e^{\frac{\pi i}{2} B_{2,2}(z; \omega_1, \omega_2)} \prod_{i=0}^{\infty} \frac{(1 - e^{2\pi i \frac{z}{\omega_2}} \tilde{q} \tilde{q}^i)}{(1 - e^{2\pi i \frac{z}{\omega_1}} q^i)}, \quad (25)$$

where parameters are $\tilde{q} = e^{2\pi i \omega_1 / \omega_2}$ and $q = e^{-2\pi i \omega_2 / \omega_1}$ and the Bernoulli polynomial is

$$B_{2,2}(z; \omega_1, \omega_2) = \frac{z^2 - z(\omega_1 + \omega_2)}{\omega_1 \omega_2} + \frac{\omega_1^2 + 3\omega_1 \omega_2 + \omega_2^2}{6\omega_1 \omega_2}. \quad (26)$$

One of the several representations of this special function is the following

$$\gamma^{(2)}(z; \omega_1, \omega_2) = \exp \left(- \int_0^\infty \frac{dx}{x} \left[\frac{\sinh x(2z - \omega_1 - \omega_2)}{2 \sinh(x\omega_1) \sinh(x\omega_2)} - \frac{2z - \omega_1 - \omega_2}{2x\omega_1\omega_2} \right] \right), \quad (27)$$

where $Re(\omega_1), Re(\omega_2) > 0$ and $Re(\omega_1 + \omega_2) > Re(z) > 0$.

Reflection property

$$\gamma^{(2)}(z; \omega_1, \omega_2) \gamma^{(2)}(\omega_1 + \omega_2 - z; \omega_1, \omega_2) = 1. \quad (28)$$

The difference equation

$$\frac{\gamma^{(2)}(z + \omega_1; \omega_1, \omega_2)}{\gamma^{(2)}(z; \omega_1, \omega_2)} = 2 \sin\left(\frac{\pi z}{\omega_2}\right). \quad (29)$$

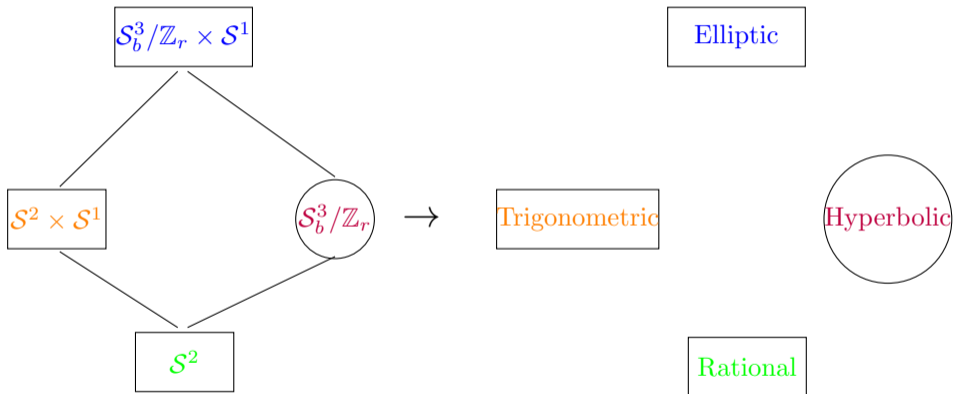
The asymptotic behaviours allows the gauge symmetry breaking

$$\lim_{z \rightarrow \infty} e^{\frac{\pi i}{2} B_{2,2}(z; \omega_1, \omega_2)} \gamma^{(2)}(z; \omega_1, \omega_2) = 1 \text{ for } \arg \omega_2 + \pi > \arg z > \arg \omega_1 \quad (30)$$

$$\lim_{z \rightarrow \infty} e^{-\frac{\pi i}{2} B_{2,2}(z; \omega_1, \omega_2)} \gamma^{(2)}(z; \omega_1, \omega_2) = 1 \text{ for } \arg \omega_2 > \arg z > \arg \omega_1 - \pi, \quad (31)$$

where $\text{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$. We use the following notation

$$\begin{aligned} \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2) &= \gamma^{(2)}(-i(a_i \pm z) - i\omega_1(u_i \pm y); -i\omega_1 r, -i\omega_1 - i\omega_2) \\ &\times \gamma^{(2)}(-i(a_i \pm z) - i\omega_2(r - (u_i \pm y)); -i\omega_2 r, -i\omega_1 - i\omega_2) \end{aligned} \quad (32)$$



Pentagon identity
[Gahramanov and Rosengren, 2014]

Knot invariant
[Spiridonov and Vartanov, 2011]







Bailey pairs
[Brünnner and Spiridonov, 2017]
[Gahramanov et al. 2022]

Quantum groups [Asperyan et al. 2022] [Bozkurt et al. 2020]	gauge/YBE correspondence [Gahramanov and Kels, 2016]	Painleve equations [Kels and Yamazaki, 2017]
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SUSY gauge theories
[Imamura and Yokoyama, 2012]

Special functions
[van de Bult, 2007]

- Does the generalized Faddeev-Volkov model corresponds to the representation of quantum groups $U_q(\mathfrak{osp}(1|r))$?
- What is the Hamiltonian of 1D spin chain via these 2D classical integrable models?
- Star-square relation and generalized star-triangle relation for $SU(2)$ models (2310.XXXXX)
- More star-star relations via two kinds of star-triangle relations, then various IRF models (2312.XXXXX)
- Decoration transformation for integrable models (2309.XXXXX)
- ...

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Thank You!