Various Disguises of the Pais-Uhlenbeck Oscillator

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Outline

Pais-Uhlenbeck Oscillator

- Lagrangian of a $1+1$-d PU oscillator[1]:

$$L = -\frac{1}{2} x \prod_{k=1}^{n} \left(\frac{d^2}{dt^2} + \lambda_k^2 \right) x, \tag{1}$$

where $\{\lambda_k\}$ are constant PU oscillator frequencies.

- $n = 1$: 1 oscillator system.
- $n = 2$ (4th order PU) case (2 oscillators)

$$L_{PU} = \frac{1}{2}\left[(x'')^2 - (\lambda_1^2 + \lambda_2^2)(x')^2 + \lambda_1^2 \lambda_2^2 x^2 \right], \tag{2}$$

$$x'''' + (\lambda_1^2 + \lambda_2^2) x'' + \lambda_1^2 \lambda_2^2 x = 0. \tag{3}$$

- Higher derivative system, energy is not positive definite.

[1] A. Pais and G.E. Uhlenbeck (1950), A. Smilga (2017)

H is a combination of harmonic oscillators with an alternating sign:

$$H = \frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1}(p_k^2 + \lambda_k^2 x_k^2). \tag{4}$$

Symmetries:

- Generic λs: n copies of Heisenberg algebra with one central extension[2].
- Odd frequencies: $\ell = \frac{2n-1}{2}$ conformal Newton-Hooke group[3].

Other works in the literature:

- Quantization was made[4],
- interactions added[5],
- generalized to odd orders[6]...

[2] K. Andrzejewski (2014)
[3] K. Andrzejewski et al. (2014)
[4] C. M. Bender and P. D. Mannheim (2008)
[5] M. Pavsic (2013)
[6] I. Masterov (2016)

Our Results

$1+1$-*d PU oscillator appears in the dynamics of several physical systems with second order eom*:

- Lower dimensional NR systems:
  1. $2+1$ dimensional exotic particle.
  2. $3+1$ dimensional Penning trap (*Nobel prize 1989*),

- $1+3$ dimensional plane gravitational waves (GWs) via *ED lift*:
  1. Circularly polarized periodic GW (*inflanatory models*),
  2. Lukash plane GW (*relevant for cosmology*),
  3. Complete pulse GW (*with a proper CKV*).

Newton-Hooke symmetry of PU oscillator yields,
i) Carroll symmetry of GWs and more,
ii) charges of exotic particle and Penning trap

Alternative formulation

Instead of (2), begin with an alternative $2+1$-d Lagrangian,

$$L = \frac{1}{2}M(x'^2 + y'^2) - \frac{M\omega}{2}(yx' - xy') + \frac{M}{2}(\Omega_+^2 x^2 + \Omega_-^2 y^2), \quad (5)$$

where M, ω and $\Omega_\pm$ are real constants.

- time-independent oscillators of mass M,
- with Coriolis force/magnetic field of strength ω,
- under anisotropic inverted potential with $\Omega_\pm$.

Simple set of planar e.o.m.:

$$x'' - \omega y' - \Omega_+^2 x = 0, \quad (6a)$$

$$y'' + \omega x' - \Omega_-^2 y = 0. \quad (6b)$$

Solutions depend on $\Omega_\pm, \omega$.

Eqns. (6) can be cast to the form

$$x'''' + (\omega^2 - \Omega_+^2 - \Omega_-^2)x'' + \Omega_+^2\Omega_-^2 x = 0, \tag{7}$$

to yield $4th$-order Pais-Uhlenbeck oscillator (2) if

$$\lambda_1^2\lambda_2^2 = \Omega_+^2\Omega_-^2, \quad \lambda_1^2 + \lambda_2^2 = \omega^2 - \Omega_+^2 - \Omega_-^2. \tag{8}$$

We find

$$\lambda_1^2 = \frac{(\omega^2 - \Omega_+^2 - \Omega_-^2) + \sqrt{\Delta}}{2}, \qquad \lambda_2^2 = \frac{(\omega^2 - \Omega_+^2 - \Omega_-^2) - \sqrt{\Delta}}{2}, \tag{9}$$

where $\Delta = (\omega^2 - \Omega_+^2 - \Omega_-^2)^2 - 4\Omega_+^2\Omega_-^2$.

Explicit relation to PU oscillator

Define kinematical momenta $\Pi_i = (\Pi_x, \Pi_y)$ from (5)

$$\Pi_i = p_i + \frac{M\omega}{2}\epsilon_{ij}x^j, \qquad p_x = \frac{\partial L}{\partial x'}, \quad p_y = \frac{\partial L}{\partial y'} \tag{10}$$

such that the symplectic structure becomes

$$H = \frac{\Pi_x^2}{2M} + \frac{\Pi_y^2}{2M} - \frac{M}{2}(\Omega_+^2 x^2 + \Omega_-^2 y^2), \tag{11a}$$

$$\sigma = d\Pi_i \wedge dx^i + M\omega dx \wedge dy. \tag{11b}$$

Chiral decomposition: New phase space coordinates $(X_+^{1,2}, X_-^{1,2})$:

$$\Pi_x = \alpha_+ X_+^2 + \alpha_- X_-^2, \quad \Pi_y = -\beta_+ X_+^1 - \beta_- X_-^1, \tag{12a}$$

$$x = X_+^1 + X_-^1, \qquad y = X_+^2 + X_-^2, \tag{12b}$$

Coefficients $\alpha_\pm, \beta_\pm$ should satisfy

$$\beta_+\beta_- = M^2\Omega_+^2, \qquad \alpha_+\alpha_- = M^2\Omega_-^2, \tag{13a}$$
$$\alpha_+ + \beta_- = M\omega, \qquad \alpha_- + \beta_+ = M\omega. \tag{13b}$$

We solve in terms of PU oscillator frequencies $\lambda_{1,2}$ (9) as

$$\beta_- = \frac{M}{\omega}(\lambda_1^2 + \Omega_+^2), \qquad \beta_+ = \frac{M\omega\Omega_+^2}{\lambda_1^2 + \Omega_+^2} \tag{14a}$$
$$\alpha_+ = \frac{M}{\omega}(\lambda_2^2 + \Omega_-^2), \qquad \alpha_- = \frac{M\omega\Omega_-^2}{\lambda_2^2 + \Omega_-^2}. \tag{14b}$$

The symplectic structure (11) is decomposed into $+$ and $-$ sectors

$$H = -\frac{M(\lambda_1^2-\lambda_2^2)}{2}\left(\frac{\lambda_2^2}{(\lambda_2^2+\Omega_-^2)}(X_+^1)^2 + \frac{\lambda_2^2+\Omega_-^2}{\omega^2}(X_+^2)^2 - \frac{\lambda_1^2}{(\lambda_1^2+\Omega_+^2)}(X_-^2)^2 - \frac{\lambda_1^2+\Omega_+^2}{\omega^2}(X_-^1)^2\right) \tag{15a}$$

$$\sigma = \frac{M(\lambda_1^2-\lambda_2^2)}{\omega}(dX_+^1 \wedge dX_+^2 - dX_-^1 \wedge dX_-^2), \tag{15b}$$

with e.o.m. of harmonic oscillators:

$$(X_+^{1,2})'' + \lambda_2^2 X_+^{1,2} = 0, \quad (X_-^{1,2})'' + \lambda_1^2 X_-^{1,2} = 0. \tag{16}$$

Return to canonical coordinates with

$$\sqrt{\frac{M(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 + \Omega_+^2)}{\omega^2}} X_-^1 = -p_1, \qquad \sqrt{\frac{M(\lambda_1^2 - \lambda_2^2)}{\lambda_1^2 + \Omega_+^2}} X_-^2 = x^1, \tag{17a}$$

$$\sqrt{\frac{M(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 + \Omega_-^2)}{\omega^2}} X_+^2 = -p_2, \qquad \sqrt{\frac{M(\lambda_1^2 - \lambda_2^2)}{\lambda_2^2 + \Omega_-^2}} X_+^1 = x^2, \tag{17b}$$

Two harmonic oscillators with a relative minus sign between them:

$$H = H_1 - H_2, \qquad H_i = \frac{1}{2}\left(p_i^2 + \lambda_i^2 (x^i)^2\right). \tag{18}$$

Solutions, symmetries etc. are well known.

A final canonical transformation[7]

$$x^1 = \frac{p_x + \lambda_1^2 v}{\lambda_1 \sqrt{(\lambda_1^2 - \lambda_2^2)}}, \quad p_1 = \frac{\lambda_1 (p_v + \lambda_2^2 x)}{\sqrt{(\lambda_1^2 - \lambda_2^2)}}, \tag{19a}$$

$$x^2 = \frac{p_v + \lambda_1^2 x}{\sqrt{(\lambda_1^2 - \lambda_2^2)}}, \quad p_2 = \frac{p_x + \lambda_2^2 v}{\sqrt{(\lambda_1^2 - \lambda_2^2)}}, \tag{19b}$$

yields Ostragadski's Hamiltonian for the PU oscillator:

$$H = H_{PU} = p_x v + \frac{p_v^2}{2} + \frac{(\lambda_1^2 + \lambda_2^2) v^2}{2} - \frac{\lambda_1^2 \lambda_2^2 x^2}{2}. \tag{20}$$

Symmetries (translation, boost and 2 accelerations) of (20) yields conserved charges for (18).

Question: Any physical system described by (5) or (11)?

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[7] A. Smilga (2017)

Exotic particle in 2+1-d

To postulate non-commutativity, just modify with θ

$$\sigma\star = \sigma + \theta d\Pi_x \wedge d\Pi_y, \tag{21a}$$

$$H = \frac{\Pi_x^2}{2M} + \frac{\Pi_y^2}{2M} - \frac{M}{2}(\Omega_+^2 x^2 + \Omega_-^2 y^2). \tag{21b}$$

Poisson brackets become

$$\{\Pi_i, \Pi_j\} = \frac{\epsilon_{ij} M\omega}{1 - M\omega\theta}, \quad \{x^i, \Pi_j\} = \frac{\delta_{ij}}{1 - M\omega\theta}, \quad \{x^i, x^j\} = \frac{\epsilon_{ij}\theta}{1 - M\omega\theta}.$$

The decomposition conditions alter as

$$\beta_+\beta_- = M^2\Omega_+^2, \qquad \alpha_+\alpha_- = M^2\Omega_-^2, \tag{22a}$$

$$\alpha_+ + \beta_- - \theta\alpha_+\beta_- = M\omega, \qquad \alpha_- + \beta_+ - \theta\alpha_-\beta_+ = M\omega, \tag{22b}$$

cf. (13). Yet, can be solved with *small* θ.

In particular, we find

$$\lambda_1^2 = \frac{\alpha_-\beta_-}{M^2(1-\theta\alpha_-)(1-\theta\beta_-)}, \quad \lambda_2^2 = \frac{\alpha_+\beta_+}{M^2(1-\theta\beta_+)(1-\theta\alpha_+)}, \tag{23}$$

and obtained the PU oscillator H with θ-dependent λs,

$$H_{PU}^{\theta} = p_x v + \frac{p_v^2}{2} + \frac{\lambda_1(\theta)^2 + \lambda_2(\theta)^2}{2} v^2 - \frac{\lambda_1(\theta)^2 \lambda_2(\theta)^2}{2} x^2. \tag{24}$$

- θ-dependent charges span exotic NH algebra with two central extensions[8],

$$\{Q_1^c, Q_1^b\} = \frac{M}{1-\theta\alpha_+}, \qquad \{Q_2^c, Q_2^b\} = \frac{M}{1-\theta\alpha_-}. \tag{25}$$

- When $\theta = 0$, the usual case is recovered.

[8] C. Duval et al. (2001)

Penning trap

The Penning trap[9] are used for
i) mass spectroscopy, ii) measuring properties of charged particles.

The Lagrangian of an ion in an ideal Penning trap is

$$L_P = \frac{1}{2}M(x'^2+y'^2+z'^2)+\frac{M\omega_c}{2}(xy'-yx')+\frac{1}{4}M\omega_z^2(x^2+y^2-2z^2).$$

$\omega_c = \frac{qB}{M}$ is cyclotron and $\omega_z = \frac{qV_0}{M}$, q is the axial frequencies.

Potential $V(x,y,z) = \frac{V_0}{4}(2z^2 - x^2 - y^2)$ satisfies $\nabla^2 V = 0$.

Equations of motion are

$$x'' - \omega_c y' - \frac{1}{2}\omega_z^2 x = 0, \tag{26a}$$

$$y'' + \omega_c x' - \frac{1}{2}\omega_z^2 y = 0, \tag{26b}$$

$$z'' + \omega_z^2 z = 0. \tag{26c}$$

[9] H. G. Dehmelt (1990)

$x - y$: particular case of (6) with $\Omega_{\pm}^2 = \frac{1}{2}\omega_z^2$ and $\omega = \omega_c$.

$z-$direction: decoupled harmonic oscillator with ω_z.

Motions in the Penning trap can be separated as[10]:

- harmonic motion along the with axial frequency ω_z,
- cyclotron motion with modified cyclotron frequency ω_+,
- the circular magnetron motion at magnetron frequency ω_-,

$\omega_{\pm}$ are defined as,

$$\omega_{\pm} = \frac{\omega_c \pm \sqrt{\omega_c^2 - 2\omega_z^2}}{2}. \qquad (27)$$

Typically in Penning traps $\omega_c > \omega_+ \gg \omega_z \gg \omega_-$. The magnetron motion energy is negative.

[10] L. S. Brown and G. Gabrielse (1986), K. Blaum (2006)

PU oscillator frequencies (9) have a direct physical meaning:

$$\lambda_1 = \omega_+, \qquad \lambda_2 = \omega_-, \qquad \lambda_3 = \omega_z. \tag{28}$$

Chiral decomposition coefficients (12)

$$\beta_- = M\omega_+ = \alpha_-, \quad \alpha_+ = M\omega_- = \beta_+, \tag{29}$$

decompose the system

$$H_P = \frac{1}{2}\left[\left(p_1^2 + \omega_+^2 (x^1)^2\right) - \left(p_2^2 + \omega_-^2 (x^2)^2\right) + \left(p_z^2 + \omega_z^2 z^2\right)\right],$$

Penning trap is a $n = 3$ *(6th order) PU oscillator.*

ED lift and GWs

- $2+1$-d NR motion can be Eisenhart-Duval lifted to a $1+3$-d Bargmann metric[11] as a *null geodesic*.
- Null projection of the $1+3$-d motions yields this NR system.
- Applied to many systems including PU oscillator[12].
- When our NR system (11) is ED lifted, we get the following $1+3$d plane-wave metric with coordinates (x, y, u, v)

$$g = dx^2 + dy^2 + 2dudv + \omega(xdy - ydx)du + (\Omega_+^2 x^2 + \Omega_-^2 y^2)du^2, \tag{30}$$

- Plane waves are models for gravitational radiation.
- Generic symmetries: 5-parameter Carroll group[13] with a homothety.

[11] L. P. Eisenhart, (1928); C. Duval et al., 1985
[12] A. Galajinsky and I. Masterov (2016)
[13] C. Duval et al (2017)

- Depending on $\omega, \Omega_\pm$, (30) can be a vacuum solution or not.
- If it is a vacuum solution, it is an exact plane GW.

Affinely parametrized $L_{geo} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$, yields

$$\ddot{x} - \omega \dot{y} \dot{u} - \Omega_+^2 x \dot{u}^2 = 0, \tag{31a}$$
$$\ddot{y} + \omega \dot{x} \dot{u} - \Omega_-^2 y \dot{u}^2 = 0, \tag{31b}$$
$$\ddot{u} = 0, \tag{31c}$$

u affine parameter and obtain (6) where u becomes NR time.

Constrain geodesic Hamiltonian to null geodesics

$$H_{geo} = \frac{g^{\mu\nu} p_\mu p_\nu}{2} \equiv 0, \qquad p_\mu = (p_x, p_y, p_u, p_v), \tag{32}$$

and get NR Hamiltonian (11) as $p_u = -H$. u is NR time, $p_v = M$ and v becomes NR action.

Correspondence with NR system allows to find *Carroll Killing vectors in terms of* $\lambda_{1,2}$ (9)

$$Y_1^c = \frac{1}{2(\lambda_2^2+\Omega_-^2)}\Big(\omega(\lambda_2^2-\Omega_-^2) y \cos\lambda_2 u - \lambda_2(\Omega_+^2-\Omega_-^2+\sqrt{\Delta}) x \sin\lambda_2 u\Big)\partial_v + \cos\lambda_2 u\partial_x - \frac{\omega\lambda_2}{\lambda_2^2+\Omega_-^2}\sin\lambda_2 u\partial_y\,, \tag{33a}$$

$$Y_2^c = \frac{1}{2(\lambda_1^2+\Omega_+^2)}\Big(-\omega(\lambda_1^2-\Omega_+^2) x \cos\lambda_1 u + \lambda_1(\Omega_+^2-\Omega_-^2+\sqrt{\Delta}) y \sin\lambda_1 u\Big)\partial_v + \frac{\omega\lambda_1}{\lambda_1^2+\Omega_+^2}\sin\lambda_1 u\partial_x + \cos\lambda_1 u\partial_y\,, \tag{33b}$$

$$Y_1^b = -\frac{\omega}{\sqrt{\Delta}}\left[\left(\frac{(\Omega_+^2-\Omega_-^2+\sqrt{\Delta})}{2\omega} x \cos\lambda_2 u + \frac{(\lambda_2^2-\Omega_-^2)}{2\lambda_2} y \sin\lambda_2 u\right)\partial_v + \frac{(\lambda_2^2+\Omega_-^2)}{\omega\lambda_2}\sin\lambda_2 u\partial_x + \cos\lambda_2 u\partial_y\right], \tag{33c}$$

$$Y_2^b = -\frac{\omega}{\sqrt{\Delta}}\left[\left(\frac{(\Omega_+^2-\Omega_-^2+\sqrt{\Delta})}{2\omega} y \cos\lambda_1 u + \frac{(\lambda_1^2-\Omega_+^2)}{2\lambda_1} x \sin\lambda_1 u\right)\partial_v + \cos\lambda_1 u\partial_x - \frac{(\lambda_1^2+\Omega_+^2)}{\omega\lambda_1}\sin\lambda_1 u\partial_y\right]. \tag{33d}$$

with

$$[Y_i^c, Y_j^b] = -\delta_{ij}\partial_v. \tag{34}$$

- (30) maximally symmetric, it has an extra isometry ∂_u.
- Natural conclusion: *Any plane wave with PU oscillator as being the NR underlying system, should be endowed with an extra symmetry.*
- CPP GW[14]: vacuum solution with an extra screw isometry,
- Lukash plane wave again vacuum solution with an extra isometry,
- Complete pulse profile[15] with an extra conformal symmetry.

[14] Stephani et al., (2003)
[15] K. Andrzejewski and S. Prencel (2019)

CPP GW

In Brinkmann coordinates (X^1, X^2, U, V), a $1+3$-d CPP GW[16] is

$$
\begin{aligned}
ds^2 &= \delta_{ij} dX^i dX^j + 2dUdV + K_{ij}(U) X^i X^j dU^2, \qquad (35) \\
K_{ij}(U) X^i X^j dU^2 &= \frac{A_0}{2}\Big(\cos(\omega U)\big((X^1)^2 - (X^2)^2\big) + 2\sin(\omega U) X^1 X^2\Big),
\end{aligned}
$$

where $i, j = 1, 2$.

- A_0 is the amplitude and ω is frequency.
- Vacuum solution of Einstein's equations.
- In addition to 5-parameter Carroll and homothety, it is endowed with a screw isometry:

$$
Y_s = \partial_U + \frac{\omega}{2}\epsilon_{ij} X^i \partial_j. \qquad (36)
$$

[16] D. Kramer et al., 2003

It hints

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\omega U/2) & \sin(\omega U/2) \\ -\sin(\omega U/2) & \cos(\omega U/2) \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \quad U = u, \quad V = v.$$

CPP GW metric (35) can be brought to the form (30) with

$$\Omega_\pm^2 = \frac{\omega^2}{4} \pm \frac{A_0}{2}, \tag{37}$$

with $\omega^2 = 2(\Omega_+^2 + \Omega_-^2)$ and $Y_s = \partial_u$.
A simple calculation of (9) gives PU oscillator frequencies

$$\Omega_+^2 = \lambda_1^2, \quad \Omega_-^2 = \lambda_2^2. \tag{38}$$

When $\omega^2 > 2A_0$, PU frequencies ordered as $\lambda_1 > \lambda_2 > 0$.
The coefficients:

$$\beta_- = \frac{2M}{\omega}\lambda_1^2, \quad \beta_+ = \frac{M\omega}{2} = \alpha_-, \quad \alpha_+ = \frac{2M}{\omega}\lambda_2^2. \tag{39}$$

PU oscillator is the underlying NR system of CPP GW.

Discussion

- PU oscillator is linked to GWs, Penning trap ...
- Symmetries of PU oscillator yields symmetries of those systems.
- Conformal symmetries related to distorted ones?
- Wider parameter range e.g., $\lambda_1 = \lambda_2$?
- Relation to the ED lift PU oscillator of Galajinsky and Masterov?
- *Exotic case*: Any condensed matter realization?
- *Kerr-Schild double copy*:
  i) Relevant null e.m. configurations (knotted?)
  ii) Duality and optical helicity, other e.m. zilches...