2d CFT/gauge theory correspondence in Argyres-Douglas points

Rubik Poghossian, in collaboration with
Francesco Fucito and Francisco Morales

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Plan

- N=2 SYM and SW prepotential
- Moduli space of instantons: ADHM construction
- Induced action
- Ω-background: Generalized partition function
- Liouville conformal block and AGT relation
- Gaiotto curve and Liouville irregular states
- Application to Argyres-Douglas theories
- Comparison with WKB and Painlevé τ-function
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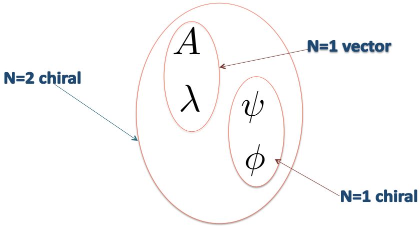
The field content and action



$$S = \int d^4x d^4\theta \, \Im\tau \, \mathrm{tr}\Psi^2$$

Scalar potential: $V \sim \mathrm{tr}[\phi, \phi^\dagger]^2$

Low energy effective action

Below Ψ includes only massless fields (i.e. those frome the Cartan of the gauge group)

$$S_{eff} = \int d^4x d^4\theta \, \Im \mathcal{F}(\Psi)$$

$\mathcal{F}$ - the Seiberg Witten prepotential
In the case of $SU(2)$

$$\mathcal{F}(\Psi) = \frac{i}{2\pi}\Psi^2 \log \frac{2\Psi^2}{e^3\Lambda^2} - \frac{i}{\pi}\sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{\Psi}\right)^{4k}\Psi^2$$

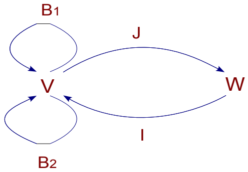
$\mathcal{F}_1 = \frac{1}{2}$, $\mathcal{F}_2 = \frac{5}{16}$, $\mathcal{F}_3 = \frac{3}{4}$, $\mathcal{F}_4 = \frac{1469}{512}, \ldots$

Moduli space of instantons, ADHM

gauge group: $U(N)$; instanton number: k; $V = \mathbb{C}^k$; $W = \mathbb{C}^N$
ADHM equations:

$$[B_1, B_2] + IJ = 0; \quad \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = \zeta$$

Equivalence relation: $(B_i, I, J) \sim (\phi B_i \phi^{-1}, \phi I, J\phi^{-1})$, $\phi \in U(k)$
Global gauge trans. : $(B_i, I, J) \to (B_i, Ig, g^{-1}J)$, $g \in U(N)$
Rotations of Euclidean space time: $(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$
$(B_i, I, J) \to (e^{i\epsilon_i} B_i, I, e^{i\epsilon_1 + i\epsilon_2} J)$,



The induced action

R.Flume, R.P., H.Storch 'arXiv:hep-th/0110240

$$\mathcal{F}_k \simeq \int_{\mathcal{M}'_k} e^{-d_x \omega},$$

$d_x \equiv d + i_x$ is an equivariant exterior derivative, i_x denotes contraction with the vector field x which generates the $U(1)$ subgroup of global gauge transformations selected by the choice of "Higgs" expectation values $\langle \phi \rangle_{cl} = diag(a_1, \ldots, a_N)$.
ω is the differential one-form

$$\omega = G(x, \bullet)$$

$G(\bullet, \bullet)$ is the natural induced metric on moduli space.

Localization to the zero locus of the vector field x

The coefficient $\mathcal{F}_k$ may be deformed into

$$\mathcal{F}_k(t) \equiv \int_{\mathcal{M}'_k} e^{-\frac{1}{t} d_x \omega}$$

Compute

$$\frac{d}{dt}\mathcal{F}_k(t) = -\frac{1}{t^2}\int_{\mathcal{M}'_k} d_x \left(\omega e^{-\frac{1}{t} d_x \omega}\right) = -\frac{1}{t^2}\int_{\mathcal{M}'_k} d \left(\omega e^{-\frac{1}{t} d_x \omega}\right).$$

The saddle point approximation is exact! There are contributions only from the points where $x = 0$. Unfortunately they are too many: in fact union of sub-manifolds of dimensions $2Nk - 4$ (c.f. $\dim \mathcal{M}'_k = 4Nk - 4$)

Incorporating space-time rotations

A wonderful way out: modify the vector field x incorporating (Euclidean) space-time rotations (parametrized by ϵ_1, ϵ_2) with the global gauge transformations (parametrized by the expectations values $a_1, \ldots, a_N$)

$$Z_k\left(a_u, \epsilon_1, \epsilon_2\right) \equiv \int_{\mathcal{M}_k} e^{-d_{\tilde{x}}\tilde{\omega}},$$

$\tilde{x}$ is the modified vector field and

$$\tilde{\omega} = G(\tilde{x}, \bullet)$$

Now we are lucky: the vector field $\tilde{x}$ has finitely many zeros!

Generalized partition function

complete localization!

$$Z_k\left(a_u,\epsilon_1,\epsilon_2\right) = \sum_{i \in \textit{fixed points}} \frac{1}{\det \mathcal{L}_{\tilde{x}}}\Bigg|_i .$$

How this is related to SW prepotential? Introduce the partition function Nekrasov 'arXiv:hep-th/0206161

$$Z(a_u,\epsilon_1,\epsilon_2,q) \equiv 1+\sum_{k=1}^{\infty} Z_k(a,\epsilon_1,\epsilon_2)q^k = e^{\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(a_u,\epsilon_1,\epsilon_2,q)}$$

$\frac{1}{\epsilon_1\epsilon_2}$ is the "voume factor" and $\mathcal{F}(a_u,0,0,q)$ coincides with the instanton part of SW prepotential.

Remarks on $\mathcal{N}=2$ SYM in Ω background

From the point of view of the initial theory above modification boils down to the consideration of the $\mathcal{N}=2$ SYM in a specific so called Ω-background specified by ϵ_1, ϵ_2, introduced in [Moor,Nekrasov,Shatashvili 'arXiv:hep-th/9712241], Losev,Nekrasov,Shatashvili 'arXiv:hep-th/9801061 to regularize the integrals over moduli space of instantons.

In Nekrasov 'arXiv:hep-th/0206161

- is shown how the partition function in this background is related to the Seiberg-Witten prepotential.
- Explicit calculation of the prepotential up to 5 instantons for the special case $h=\epsilon_1=-\epsilon_2$ is performed demonstrating that at vanishing h one exactly recovers the Seiberg-Witten curve result.

Partition function with generic ϵ_1, ϵ_2

In Flume, R.P. 'arXiv:hep-th/0208176

- a closed combinatorial formula which allows to calculate the partition function for generic ϵ_1, ϵ_2 was found. The partition function is represented as a sum over arrays of Young diagrams with total number of boxes equal to the number of instantons.
- The partition function with generic ϵ_1, ϵ_2 is essential also from the point of view of the AGT duality Alday,Gaiotto,Tachikawa ' arXiv:0906.3219 relating this partition function to the conformal blocks in 2d Conformal Field Theory.

Partition function with generic ϵ_2 and $\epsilon_1 = 0$

- In a parallel very interesting development Nekrasov and Shatashvily in 'arXiv:0908.4052 show that when $\epsilon_2 = 0$ the prepotential is related to the quantum integrable many body systems and leads to the notion of "quantum" Seiberg-Witten curve R.P. 'arXiv:1006.4822
- Note one more point which makes the investigation of $\epsilon_2 = 0$ case interesting: AGT relation this limit is related to the quasi-classical $c \to \infty$ "heavy" limit of conformal blocks in 2d CFT.

Classification of the fixed points

As we saw it is useful to generalize the Seiberg-Witten prepotential including into the game besides unbroken global gauge transformations also the space time rotations which allowed to localize instanton contributions around finite number of fixed points.

For the gauge group $U(N)$ the fixed points are in 1-1 correspondence with the arrays of Young tableau $\vec{Y} = (Y_1, ..., Y_N)$ with total number of boxes $|\vec{Y}|$ being equal to the instanton charge k.

Demonstration: arm and leg lengths

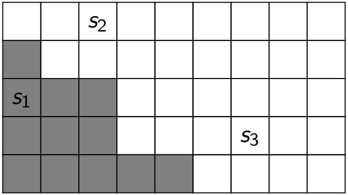


Figure: Arm and leg length with respect to a Young diagram (pictured in gray): $A(s_1) = 1,\ L(s_1) = 2,\ A(s_2) = -2,\ L(s_2) = -3,\ A(s_3) = -2,\ L(s_3) = -4.$

The character

At a fixed point the tangent space of the moduli space of instantons decomposes into sum of (complex) one dimensional irreducible representations of the Cartan subgroup of $U(N) \times O(4)$

[R.Flume, R.P. '02]

$$\chi = \sum_{\alpha,\beta=1}^{N} \frac{e_\alpha}{e_\beta} \left\{ \sum_{s\in Y_\alpha} T_1^{-L_{Y_\beta}(s)} T_2^{A_{Y_\alpha}(s)+1} + \sum_{s\in Y_\beta} T_1^{L_{Y_\alpha}(s)+1} T_2^{-A_{Y_\beta}(s)} \right\}$$

where $(e_1, ..., e_N) = (e^{ia_1}, \ldots, e^{ia_N}) \in U(1)^N \subset U(N)$ and $(T_1, T_2) = (e^{i\epsilon_1}, e^{i\epsilon_2}) \in U(1)^2 \subset O(4)$, $L_Y(s)$ $(A_Y(s))$ is the distance of the right (top) edge of the box s from the limiting polygonal curve of the Young tableaux Y in horizontal (vertical) direction taken with plus sign if $s \in Y$ and with minus sign otherwise.

Determinant of $\tilde{x}$

One-dimensional subgroups of the $N+2$ dimensional torus are parametrized by $a_1, \ldots, a_N$ and ϵ_1, ϵ_2. From the physical point of view a_α are the vacuum expectation values of the complex scalar of the $\mathcal{N}=2$ gauge multiplet. The parameters of the Ω-background ϵ_1, ϵ_2 as we saw are related to space time rotations. The contribution of a fixed point to the Nekrasov partition function in the basic $\mathcal{N}=2$ case without extra hypermultiplets is simply the inverse determinant of the vector field action on the tangent space. All the eigenvalues of this vector field can be directly read off from the character formula.

Contribution of the gauge multiplet

As a result [Flume, R.P. 'arXiv:hep-th/0208176]

$$P_{gauge}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \frac{1}{E_{\alpha,\beta}(s)(\epsilon - E_{\alpha,\beta}(s))},$$

where

$$E_{\alpha,\beta} = a_\alpha - a_\beta - \epsilon_1 l_{Y_\beta}(s) + \epsilon_2 (a_{Y_\alpha}(s) + 1)$$

In general the theory may include "matter" hypermultiplets in various representations of the gauge group. In that case one should multiply the gauge multiplet contribution by another factor P_{matter}.

The "matter" part

The respective matter factors read [Bruzzo, Fucito, Morales, Tanzini '03]

$$P_{antifund}(\vec{Y}) = \prod_{\ell=1}^{f} \prod_{\alpha=1}^{N} \prod_{s_\alpha \in Y_\alpha} (\phi_{\alpha,s_\alpha} + m_\ell)$$

$$P_{adj}(\vec{Y}) = \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} (E_{\alpha,\beta}(s) - M)(\epsilon - E_{\alpha,\beta}(s) - M),$$

where m_ℓ, M are the masses of the hypermultiplets, $\epsilon = \epsilon_1 + \epsilon_2$,

$$\phi_{\alpha,s_\alpha} = a_\alpha + (i_{s_\alpha} - 1)\epsilon_1 + (j_{s_\alpha} - 1)\epsilon_2$$

and i_{s_α}, j_{s_α} are the numbers of the column and the row of the tableaux Y_α where the box s_α is located.

The partition function of the theory with matter hyper-multiplets

Finally the instanton part of generalized partition function is:

$$Z_{inst} = \sum_{\vec{Y}} q^{|\vec{Y}|} P_{gauge}(\vec{Y}) P_{matter}(\vec{Y})$$

$q = e^{2\pi i \tau_g}$, with τ_g the usual gauge theory coupling.

Example: 1-instanton computation for pure $SU(2)$

1 instanton: two fixed points

- $$\begin{aligned} \chi(\square,\bullet) &= T_1 + T_2 + \frac{e_2}{e_1} T_1 T_2 + \frac{e_1}{e_2} \\ \det(\square,\bullet) &= \epsilon_1\epsilon_2(a_2 - a_1 + \epsilon_1 + \epsilon_2)(a_1 - a_2) \end{aligned}$$

- $$\begin{aligned} \chi(\bullet,\square) &= \frac{e_2}{e_1} + \frac{e_1}{e_2} T_1 T_2 + T_1 + T_2 \\ \det(\bullet,\square) &= (a_2 - a_1)(a_1 - a_2 + \epsilon_1 + \epsilon_2)\epsilon_1\epsilon_2 \end{aligned}$$

Hence

$$Z_1 = \frac{1}{\det(\square,\bullet)} + \frac{1}{\det(\bullet,\square)} = \frac{2}{\epsilon_1\epsilon_2((a_1 - a_2)^2 - (\epsilon_1 + \epsilon_2)^2)}$$

Hence for prepotential we get

$$\mathcal{F} = \epsilon_1\epsilon_2 \log(1 + qZ_1 + O(q)^2) = \frac{2}{((a_1 - a_2)^2 - (\epsilon_1 + \epsilon_2)^2)} + O(q)^2$$

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Liouville theory

Action

$$S = \frac{1}{4\pi}\int (\partial_a\varphi\partial^a\varphi + 4\pi\mu e^{2b\varphi})d^2x$$

This is a CFT endowed with holomorphic $T(z)$

$$T(z) = \sum_{n=-\infty}^{\infty}\frac{L_n}{z^{n+2}}$$

L_n-s are the Virasoro generators satisfying

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

The central charge $c = 1 + 6Q^2$, where $Q = (b + 1/b)$. Primary fields are the exponentials $V_\alpha = \exp 2\alpha\varphi$ with dimension $\Delta_\alpha = \alpha(Q-\alpha)$.

4-point correlation functions and conformal blocks

The main objects of interest are the 4-point correlation functions

$$\langle V_{\alpha_1}(\infty) V_{\alpha_2}(1) V_{\alpha_3}(q) V_{\alpha_4}(0) \rangle$$

Denote its holomorphic building block with fixed s-channel intermediate dimension Δ_α by $G(\alpha, \alpha_i; q)$.
AGT correspondence states:

$$Z(a_u^{(\ell)}, \epsilon_1, \epsilon_2; q) = q^{\Delta - \Delta_3 - \Delta_4} (1-q)^{2\alpha_2 (Q-\alpha_3)} G(\alpha, \alpha_i; q)$$

where gauge and CFT parameters are related by a simple map.

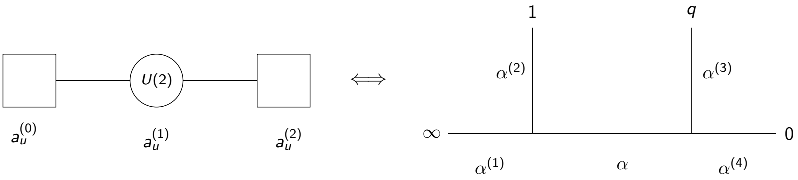

Figure: On the left: the quiver diagram for the conformal $U(2)$ gauge theory. On the right: the diagram of the conformal block for the dual Liouville field theory.

AGT dictionary

$$\frac{Q}{2}-\alpha_1 = \frac{1}{2}(a_1^{(0)}-a_2^{(0)})\,; \qquad \alpha_2=-\frac{1}{2}(a_1^{(0)}+a_2^{(0)})\,;$$
$$\alpha_3 = \frac{1}{2}(a_1^{(2)}+a_2^{(2)})\,; \qquad \frac{Q}{2}-\alpha_4=\frac{1}{2}(a_1^{(2)}-a_2^{(2)})\,;$$
$$\frac{Q}{2}-\alpha = \frac{1}{2}(a_1^{(1)}-a_2^{(1)})\equiv a\,; \qquad \epsilon=\epsilon_1+\epsilon_2=Q\,;$$
$$\epsilon_1 = b\,; \qquad \epsilon_2=b^{-1}\,;$$

Why this is true?
Two "explanations"

- Physical: Through M-theory engineering of a 6d theory on $R^5 \times S_{2,4}$
- Algebraic geometry: There is a natural action of the Virasoro (W) algebra on the moduli space of instantons

Generalization

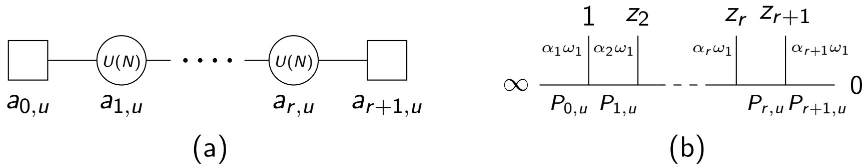

Figure: (a) The quiver diagram for the conformal linear quiver $U(N)$ gauge theory: r circles stand for gauge multiplets; two squares represent N anti-fundamental (on the left edge) and N fundamental (the right edge) hypermultiplets; the lines connecting adjacent circles are the bi-fundamentals. (b) The AGT dual conformal block of the Toda field theory.

How to generalize? Consider $\mathcal{N}=2$ $U(n)^r$ quiver SYM theory with n fundamental, n anti-fundamental and $r-1$ bi-fundamental hypers in Ω-background. The instanton part of the partition function

$$Z_{inst}=\sum_{Y_u^{(\alpha)}} F_{Y_u^{(\alpha)}} \prod_{\alpha=1}^{r}\prod_{u=1}^{n} q_\alpha^{|Y_u^{(\alpha)}|},$$

where $Y_u^{(\alpha)}$ denote $r\times n$ Young diagrams, $||$ is the number of boxes and q_α are couplings. The coefficients:

$$F_{Y_u^{(\alpha)}}=\prod_{u,v=1}^{n}\frac{\prod_{\alpha=0}^{r} Z_{bf}(a_u^{(\alpha)},Y_u^{(\alpha)}\mid a_v^{(\alpha+1)},Y_v^{(\alpha+1)})}{\prod_{\alpha=1}^{r} Z_{bf}(a_u^{(\alpha)},Y_u^{(\alpha)}\mid a_v^{(\alpha)},Y_v^{(\alpha)})},$$

where

$$Z_{bf}(a,\lambda\mid b,\mu)=$$
$$\prod_{s\in\lambda}\left(a-b-\epsilon_1 L_\mu(s)+\epsilon_2(1+A_\lambda(s))\right)\ \prod_{s\in\mu}\left(a-b+\epsilon_1(1+L_\lambda(s))-\epsilon_2 A_\mu(s)\right).$$

A shortcut from 2d CFT to 4d gauge theory

AGT relates CFT block with 4d partition function, but the latter is not the only object of interest. In fact SW curve encodes more information.
The case of $SU(2)$ with $N_f = 4$
Consider normalized expectation value of $T(z)$

$$\phi_2 = -\frac{\langle\alpha_1|V_{\alpha_2}(1)T(z)V_{\alpha_3}(q)|\alpha_4\rangle}{\langle\alpha_1|V_{\alpha_2}(1)V_{\alpha_3}(q)|\alpha_4\rangle}$$

then $\sqrt{\phi_2}\,dz$ can be identified with SW differential λ_{SW}. Even more, the differential equation $(\epsilon_2^2\partial_z^2 + \phi_2)\psi(z) = 0$ coincides with Fourier transform of T-Q equation we encountered in NS limit.

Question

This final part of my talk is based on [Fucito, Morales, R.P., arXiv:2306.05127]

How to construct other gauge theories, say with less number of flavors or Argyres-Douglas theories which are strongly coupled, non-Lagrangian, with mutually non-local light electric and magnetic degrees of freedom?

It happens that one should involve irregular states of Liouville theory.

Irregular states in Lioville theory

Example: rank-1 irregular state
Consider

$$\text{const}\, x^{2\alpha_1\alpha_2} V_{\alpha_2}(x)|\alpha_1\rangle|_\beta$$

the leading singularity as $x \to 0$:

$$\beta(Q-\beta) - \alpha_1(Q-\alpha_1) - \alpha_2(Q-\alpha_2) + 2\alpha_1\alpha_2$$
$$= \beta(Q-\beta) + (\alpha_1+\alpha_2)^2 - Q(\alpha_1+\alpha_2)$$

Sending $\alpha_{1,2} \to \infty$, keeping $c_0 = \alpha_1 + \alpha_2$ and $c_1 = \alpha_1 x$ and choosing const $\sim \alpha_1^{c_0(Q-c_0)}$, we get a finite limiting state denoted as $irr^{(1)(c_0,c_1;\beta)}$.

Similarly one can collide more primaries to get higher rank irregular states [Gaiotto, Teschner arXiv:1203.1052]

Rank 2 irregular state and the H_2 Argyres-Douglas theory

the state $I^{(2)}(c_0, c_1, c_2; \beta_0, \beta_1)$ can be defined by equations

$$
\begin{aligned}
L_k I^{(2)} &= \mathcal{L}_k I^{(2)} \quad , \qquad k = 0, \dots 4 \\
L_k I^{(2)} &= 0 \quad , \qquad k > 4
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_0 &= c_0(Q - c_0) + c_1 \partial_{c_1} + 2c_2 \partial_{c_2} \\
\mathcal{L}_1 &= 2c_1(Q - c_0) + c_2 \partial_{c_1} \\
\mathcal{L}_2 &= -c_1^2 + c_2(3Q - 2c_0) \, ; \quad \mathcal{L}_3 = -2c_1 c_2 \, ; \quad \mathcal{L}_4 = -c_2^2
\end{aligned}
$$

The irregular block of is given by scalar product

$$Z_{\mathcal{H}_2} = \langle \beta_0 | I^{(2)} \rangle$$

We define

$$\begin{aligned}\phi_2(z) &= -\frac{\langle \beta_0 | T(z) | I^{(2)} \rangle}{\langle \beta_0 | I^{(2)} \rangle} = \\ &- \frac{\Delta_{\beta_0}}{z^2} + \frac{2v}{z^3} + \frac{c_1^2 + c_2(2c_0 - 3Q)}{z^4} + \frac{2c_1 c_2}{z^5} + \frac{c_2^2}{z^6}\end{aligned}$$

where (the analogue of generalized Matone relation [Flume,Fucito,Morales,R.P.:hep-th/0403057] for the case of AD-theory)

$$v = -\frac{c_2}{2}\,\partial_{c_1} \log Z_{\mathcal{H}_2} + c_1(c_0 - Q)$$

The partition function can be computed in the limit $c_2 \to 0$ using expansion of I_2 in terms of I_1

$$|I^{(2)}\rangle = c_1^{\nu_1} c_2^{\nu_2} e^{(c_0-\beta_1)\frac{c_1^2}{c_2}} \left[1+c_2\left(\frac{\nu_3}{c_1^2}+\frac{(2c_0-3\beta_1)}{c_1}\partial_{c_1}+\frac{L_{-1}}{2c_1}\right)+\ldots\right] |I^{(1)}(\beta_1, c_1,)\rangle$$

with

$$\begin{aligned} \nu_1 &= 2(c_0-\beta_1)(Q-\beta_1) \quad , \quad \nu_2 = \tfrac{1}{2}(\beta_1 - c_0)(3Q-3\beta_1-c_0) \\ \nu_3 &= \tfrac{1}{2}(c_0+3Q-3\beta_1)(c_0-\beta_1)(Q-\beta_1) \end{aligned}$$

All coefficients are found recursively using definition of irregular states. For $I^{(1)}$ explicitly:

$$L_0 I^{(1)} = (\Delta_{\beta_1} + c_1\partial_{c_1}) I^{(1)}\,;\; L_1 I^{(1)} = 2c_1(Q-c_0) I^{(1)}\,;\; L_2 I^{(1)} == -c_1^2 I^{(1)}$$

Computing matrix elements we get $Z_{\mathcal{H}_2} = Z_{\mathcal{H}_2,\mathrm{tree}} Z_{\mathcal{H}_2,\mathrm{inst}}$ with

$$Z_{\mathcal{H}_2\mathrm{tree}} = c_1^{\nu_1+\Delta_{\beta_0}-\Delta_{\beta_1}} c_2^{\nu_2}\, e^{(c_0-\beta_1)\frac{c_1^2}{c_2}}$$

$$Z_{\mathcal{H}_2\mathrm{inst}} = 1+\frac{c_2}{2c_1^2}\left[2\nu_3+(2c_0-3\beta_1)(\beta_1-\beta_0)(\beta_1+\beta_0-Q)\right]+\ldots$$

Higher powers of c_2 can be calculated with more efforts.
Checking against Seiberg-Witten analysis:
go from CFT variables to gauge theory ones using the map
$(s = \epsilon_1+\epsilon_2$ and $p = \epsilon_1\epsilon_2)$

$$\hat{\phi}_2(z) = p\phi_2(z)$$

$$Q = \frac{s}{\sqrt{p}}\,; \quad c_0 = \frac{2\hat{c}_0+3s}{2\sqrt{p}}\,; \quad c_i = \frac{\hat{c}_i}{\sqrt{p}}\,,\ i=1,2\,;$$

$$\beta_0 = \frac{s-2M}{2\sqrt{p}}\,; \quad \beta_1 = \frac{s+a}{\sqrt{p}}\,; \quad v = \frac{\hat{u}}{p}$$

If Ω-background is absent ($\epsilon_{1,2} = 0$) we get SW curve of H_2 Argyres-Douglas theory

$$\widehat{\phi}_2(z) = \frac{M^2}{z^2} + \frac{2\hat{u}}{z^3} + \frac{\hat{c}_1^2 + 2\hat{c}_0\hat{c}_2}{z^4} + \frac{2\hat{c}_1\hat{c}_2}{z^5} + \frac{\hat{c}_2^2}{z^6}$$

and

$$a = \frac{1}{2\pi i}\oint_{z=0}\sqrt{\widehat{\phi}_2(z)}dz = -\frac{\hat{c}_2\left(M^2\hat{c}_1^2 + 2\hat{c}_0\hat{c}_1\hat{u} - 3\hat{u}^2\right)}{2\hat{c}_1^4} + \frac{\hat{u}}{\hat{c}_1} + \dots$$

Inverting for u one finds

$$\hat{u} = a\hat{c}_1 - \frac{\hat{c}_2\left(3a^2 - 2a\hat{c}_0 - M^2\right)}{2\hat{c}_1} + \dots$$

This matches the results for v obtained from CFT computation:

$$v = -2\beta_1 c_1 - \frac{c_2\left(\beta_0^2 - 3\beta_1^2 + 2\beta_1 c_0\right)}{c_1} + \dots$$

in view of identification rules.

Comparison with gauge theory side

In gauge theory side we have computed ϵ corrections using
-holomorphic anomaly method [Huang,Kashani-Poor,Klemm, arXiv:1109.5728] for arbitrary ϵ
-WKB in NS limit [Mironov,Morozov, arxiv:0911.2396;Fucito, Morales,Pacifici,RP, arXiv:1103.4495; Morales,Fucito,RP,arXiv:2306.05127]
-Correspondence with Painlevé IV τ-function [Bonelli,Lisovyy,Maruyoshi,Sciarappa,Tanzini,arXiv:1612.06235] valid when $\epsilon_1 + \epsilon_2 = 0$
We get perfect agreement in all cases.

Painlevé gauge correspondence

The σ-form of Penlevé IV

$$\ddot{\sigma}^2 = (t\dot{\sigma} - \sigma)^2 - 4\dot{\sigma}\left(\dot{\sigma} - 2\theta_s\right)\left(\dot{\sigma} - 2\theta_t\right)$$

The τfunction is defined by $\tau(t) = \frac{d}{dt}\log\sigma(\tau)$

$$\tau(t) = x^{-\frac{1}{4}+2(\theta_s^2-\theta_s\theta_t+\theta_t^2)} \sum_{n\in\mathbb{Z}} e^{in\rho}\mathcal{G}(\nu+n,x)\,; \qquad s = \frac{t^2}{2\sqrt{3}}$$

$$\mathcal{G}(\nu,x) = C(\nu,x)\left[1 + \sum_{k=1}^{\infty} \frac{D_k(\nu)}{x^k}\right]$$

$$C(\nu,x) = (2\pi)^{-\frac{\nu}{2}} e^{\frac{x^2}{9}+i\nu x-\frac{i\pi\nu^2}{4}+\frac{2s(\theta_s+\theta_t)}{\sqrt{3}}} x^{\frac{1}{12}-\frac{\nu^2}{2}} 6^{-\frac{\nu^2}{2}} G(1+\nu)$$

where $G(1+\nu)$ is Barnes G-function and the parameters ν, ρ are related to Stokes multipliers

The claim is that $\mathcal{G}(\nu, x)$, up to an x-independent factor coincides with partition function of $\mathcal{H}_2$ theory, provided one identifies

$$x^{-1} = \frac{c_2}{i c_1^2}\,, \quad c_0 = i(\theta_s + \theta_t)\,, \quad \beta_0 = i(\theta_t - \theta_s)\,, \quad \beta_1 = \frac{i(\theta_s + \theta_t - 3\nu)}{3}$$

Both irregular state computation and holomorphic anomaly relation confirm this conjecture.
Similar analysis has been performed also for the case of $\mathcal{H}_3$ theory which is related to rank 3 irregular state.

THANKS