One point matrix of the modular transformation in $N = 1$ supersymmetric Liouville field theory.

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Fusion matrix

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{n,-m} + \text{right copy}$$

Bulk OPE has the form

$$\Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2) = \sum_k \frac{C_{ij}^k}{|z_1-z_2|^{\Delta_i+\Delta_j-\Delta_k}}\Phi_k(z_2,\bar{z}_2) + \dots .$$

By the usual arguments we have for 4-point correlation function $\langle \Phi_i(\infty)\Phi_k(1)\Phi_j(x,\bar{x})\Phi_l(0)\rangle$ in s and t channels

$$\sum_p C_{jl}^p C_{kp}^i \mathcal{F}_p^s \left[\begin{array}{cc} k & j \\ i & l \end{array}\right](x) \bar{\mathcal{F}}_p^s \left[\begin{array}{cc} k & j \\ i & l \end{array}\right](\bar{x})$$

$$\sum_q C_{kj}^q C_{ql}^i \mathcal{F}_q^t \left[\begin{array}{cc} k & j \\ i & l \end{array}\right](x) \bar{\mathcal{F}}_q^t \left[\begin{array}{cc} k & j \\ i & l \end{array}\right](\bar{x}),$$

where $\mathcal{F}^s_p \left[\begin{array}{cc} k & j \\ i & l \end{array} \right](x)$ and $\mathcal{F}^t_q \left[\begin{array}{cc} k & j \\ i & l \end{array} \right](x)$ are s and t channels conformal blocks correspondingly. Conformal blocks in s and t channels are related by the fusing matrix

$$\mathcal{F}^s_p \left[\begin{array}{cc} k & j \\ i & l \end{array} \right](x) = \sum_q F_{p,q} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] \mathcal{F}^t_q \left[\begin{array}{cc} k & j \\ i & l \end{array} \right](x)\,, \qquad (0.1)$$

and hence one has:

$$\sum_p C^p_{jl} C^i_{kp} F_{pq} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] F^*_{pq} \left[\begin{array}{cc} k & j \\ i & l \end{array} \right] = \qquad (0.2)$$

$$C^q_{kj} C^i_{ql}\,.$$

Modular matrix for the 1-point blocks

In terms of the CFT on the complex plane the 1-point correlation function of ϕ_λ on a torus with modular parameter τ takes the form:

$$\langle \phi_\lambda \rangle = \mathrm{Tr}\left(e^{-(\Im\mathfrak{m}\tau)2\pi\hat{H}+i(\Re\mathfrak{e}\tau)2\pi\hat{P}} \phi_\lambda(1,1) \right) = (q\bar{q})^{-\frac{c}{24}} \mathrm{Tr}\left(q^{L_0}\bar{q}^{\bar{L}_0} \phi_\lambda(1,1) \right)$$

where τ is the torus modular parameter and

$$\hat{H} \;=\; L_0 + \bar{L}_0 - \frac{c}{12}, \qquad \hat{P} = L_0 - \bar{L}_0, \qquad q = e^{2\pi i\tau}.$$

One can write the 1-point correlation function on the torus in the following form:

$$\langle \phi_\lambda \rangle = \sum_{(\Delta,\bar{\Delta})} \mathcal{F}^{\lambda}_{\Delta}(q)\, \bar{\mathcal{F}}^{\lambda}_{\Delta}(\bar{q})\, C^{\Delta}_{\lambda\Delta}\ .$$

The modular matrix for the 1-point blocks on the torus defined by

$$\mathcal{F}^{\lambda}_{\lambda_s}(q(\tau)) \;=\; (-i\tau)^{-\Delta_\lambda} \sum_{\lambda_t} \mathsf{S}_{\lambda_s\lambda_t}(\lambda)\, \mathcal{F}^{\lambda}_{\lambda_t}\left(q\left(-\tfrac{1}{\tau}\right)\right) \quad (0.3)$$

The S-matrix satisfies the following Moore-Seiberg duality identity:

$$S_{\beta_1\beta_2}(\beta_3) \sum_{\beta_4} F_{\beta_3,\beta_4}\left[\begin{array}{cc} \beta_2 & \alpha_1 \\ \beta_2 & \alpha_2 \end{array}\right] e^{-2\pi i(\Delta_{\beta_4}-\Delta_{\beta_2})} F_{\beta_4,\beta_5}\left[\begin{array}{cc} \alpha_2 & \alpha_1 \\ \beta_2 & \beta_2 \end{array}\right] =$$

$$\sum_{\beta_6} F_{\beta_3,\beta_6}\left[\begin{array}{cc} \beta_1 & \alpha_1 \\ \beta_1 & \alpha_2 \end{array}\right] e^{-\pi i(\Delta_{\alpha_1}+\Delta_{\alpha_2}-\Delta_{\beta_5})} F_{\beta_1,\beta_5}\left[\begin{array}{cc} \alpha_1 & \alpha_2 \\ \beta_6 & \beta_6 \end{array}\right] S_{\beta_6\beta_2}(\beta_5)$$

Setting here $\beta_1 = \beta_3 = 0$ one can obtain an explicit expression of the S-matrix in terms of the fusion matrix :

$$S_{0\beta_2}\sum_{\beta_4} F_{0,\beta_4}\left[\begin{array}{cc}\beta_2 & \alpha\\ \beta_2 & \alpha\end{array}\right] e^{2\pi i\Delta_{\beta_4}} F_{\beta_4,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \beta_2 & \beta_2\end{array}\right] \quad (0.4)$$

$$= e^{\pi i(2\Delta_\alpha+2\Delta_{\beta_2})} F_{0,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \alpha & \alpha\end{array}\right] S_{\alpha\beta_2}(\beta_5)$$

Liouville field theory

Let us review basic facts on the Liouville field theory. Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \,, \tag{0.5}$$

where R is associated curvature. This theory is conformal invariant if the coupling constant b is related with the background charge Q as

$$Q = b + \frac{1}{b} \,. \tag{0.6}$$

The symmetry algebra of this conformal field theory is the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c_L}{12}(n^3 - n)\delta_{n,-m} \tag{0.7}$$

with the central charge

$$c_L = 1 + 6Q^2\,. \tag{0.8}$$

Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q-\alpha)\,. \tag{0.9}$$

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right] = \frac{\mathcal{N}(\alpha_4,\alpha_3,\alpha_s)\mathcal{N}(\alpha_s,\alpha_2,\alpha_t)}{\mathcal{N}(\alpha_4,\alpha_t,\alpha_1)\mathcal{N}(\alpha_t,\alpha_3,\alpha_2)}\left\{\begin{array}{cc|c}\alpha_2 & \alpha_1 & \alpha_s\\ \alpha_4 & \alpha_3 & \alpha_t\end{array}\right\}_b \tag{0.10}$$

6j-symbols

6j-symbols of the Faddeev modular quantum double: $U_q(sl(2,\mathbb{R}) \oplus U_{\tilde{q}}(sl(2,\mathbb{R}),\ q = e^{\pi i b^2}$ and $\tilde{q} = e^{\pi i b^{-2}}$.

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix} \right\}_b = \frac{S_b(\alpha_s+\alpha_2-\alpha_1)S_b(\alpha_1+\alpha_t-\alpha_4)}{S_b(\alpha_t+\alpha_2-\alpha_3)S_b(\alpha_3+\alpha_s-\alpha_4)} |S_b(2\alpha_t)|^2 J_h(\beta_a^\circ, \gamma_a^\circ)$$

$$J_h(\beta_a^\circ, \gamma_a^\circ) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} S_b(z+\gamma_a^\circ) S_b(-z+\beta_a^\circ) \frac{dz}{i}$$

$$S_b(x) = \gamma^{(2)}(x, b, b^{-1}) \qquad (0.11)$$

where $\gamma^{(2)}(x, \omega_1, \omega_2)$ is hyperbolic gamma function.

Ponsot-Teschner parametrization

$$\begin{aligned}
\gamma_1^\circ &= -Q/2+\alpha_3-\alpha_4\,, & \beta_1^\circ &= Q/2+\alpha_s\,,\\
\gamma_2^\circ &= -Q/2+\alpha_1-\alpha_2\,, & \beta_2^\circ &= Q/2-\alpha_t+\alpha_4+\alpha_2\,,\\
\gamma_3^\circ &= Q/2-\alpha_3-\alpha_4\,, & \beta_3^\circ &= -Q/2+\alpha_t+\alpha_4+\alpha_2\,,\\
\gamma_4^\circ &= Q/2-\alpha_1-\alpha_2\,, & \beta_4^\circ &= 3Q/2-\alpha_s\,.
\end{aligned} \tag{0.12}$$

$$\sum_{a=1}^{4}(\gamma_a^\circ+\beta_a^\circ)=2Q\,. \tag{0.13}$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the integral representation

$$\gamma^{(2)}(y;\omega_1,\omega_2) = \exp\left(-\int_0^\infty \left(\frac{\sinh(2y-\omega_1-\omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)} - \frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2 x}\right)\right)$$

and obeys the equations:

$$\frac{\gamma^{(2)}(y+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_2}, \quad \frac{\gamma^{(2)}(y+\omega_2;\omega_1,\omega_2)}{\gamma^{(2)}(y;\omega_1,\omega_2)} = 2\sin\frac{\pi y}{\omega_1}.$$

The function $\gamma^{(2)}(y;\omega_1,\omega_2)$ has the following asymptotics :

$$\lim_{y\to\infty} e^{\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2) = 1,\ \arg\omega_1 < \arg y < \arg\omega_2+\pi,$$

$$\lim_{y\to\infty} e^{-\frac{i\pi}{2}B_{2,2}(y;\omega_1,\omega_2)}\gamma^{(2)}(y;\omega_1,\omega_2) = 1,\ \arg\omega_1-\pi < \arg y < \arg\omega_2,$$

where $B_{2,2}(y;\omega_1,\omega_2)$ is the second order Bernoulli polynomial:

$$B_{2,2}(y;\omega_1,\omega_2) = \frac{y^2}{\omega_1\omega_2} - \frac{y}{\omega_1} - \frac{y}{\omega_2} + \frac{1}{6}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right) + \frac{1}{2}.$$

$$S_{0\beta_2}\int d\beta_4 F_{0,\beta_4}\left[\begin{array}{cc}\beta_2 & \alpha\\ \beta_2 & \alpha\end{array}\right] e^{2\pi i\Delta_{\beta_4}} F_{\beta_4,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \beta_2 & \beta_2\end{array}\right]$$

$$= e^{\pi i(2\Delta_\alpha+2\Delta_{\beta_2})} F_{0,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \alpha & \alpha\end{array}\right] S_{\alpha\beta_2}(\beta_5)$$

$$\sum \mu_k = b + b^{-1} = Q \tag{0.14}$$

$$\frac{1}{2}\int_{-i\infty}^{i\infty} \frac{\prod_1^6 S_b(\mu_k+z)S_b(\mu_k-z)}{S_b(2z)S_b(-2z)}\frac{dz}{i} = \prod_{1\leq a<b\leq 6} S_b(\mu_a+\mu_b) \tag{0.15}$$

$$\mu_k = f_k + i\nu \quad k=1,2,3 \quad \mu_k = g_k - i\nu \quad k=4,5,6 \quad z\to z-i\nu \tag{0.16}$$

$$\int \frac{dx}{i}\prod_{i=1}^{3} S_b(x+f_i)S_b(-x+g_i) = \prod_{i,j=1} S_b(f_i+g_j) \tag{0.17}$$

$$\sum_i (f_i+g_i) = Q \tag{0.18}$$

Hyperbolic E1

$$\sum_{i=1}^{8} \mu_i = 2Q \tag{0.19}$$

$$I_h(\mu_i) = \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^{8} S_b(\mu_i \pm z)}{S_b(\pm 2z)} dz \tag{0.20}$$

$$I_h(\mu_i) = \prod_{1\le i<j\le 4} S_b(\mu_i + \mu_j) \prod_{5\le i<j\le 8} S_b(\mu_i + \mu_j) I_h(\nu_i) \tag{0.21}$$

$$\xi = \frac{1}{2}(Q - \sum_{i=1}^{4} \mu_i) \tag{0.22}$$

$$\nu_i = \mu_i + \xi \quad i = 1, 2, 3, 4 \quad \nu_i = \mu_i - \xi \quad i = 5, 6, 7, 8 \tag{0.23}$$

$$S^{SL}_{\beta_1\beta_2}(\alpha_0)=\frac{\mathcal{N}(\beta_1,\alpha_0,\beta_1)}{\mathcal{N}(\beta_2,\alpha_0,\beta_2)}S_{\beta_1\beta_2}(\alpha_0) \qquad (0.24)$$

$$S_{\beta_1\beta_2}(\alpha_0)=S_{0\beta_2}\frac{e^{\frac{i\pi}{2}\Delta_{\alpha_0}}}{S_b(\alpha_0)}\int_{-i\infty}^{i\infty}\frac{dt}{i}e^{2\pi it(-2\beta_1+Q)}$$
$$\times S_b(\beta_2+\alpha_0/2-Q/2+t)S_b(\beta_2+\alpha_0/2-Q/2-t)\times$$
$$S_b(-\beta_2+\alpha_0/2+Q/2-t)S_b(-\beta_2+\alpha_0/2+Q/2+t)$$

N=1 Super Liouville field theory

Let us review basic facts on the $N=1$ Super Liouville field theory. $N=1$ super Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L}=\frac{1}{2\pi}g_{ab}\partial_a\varphi\partial_b\varphi+\frac{1}{2\pi}(\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi})+2i\mu b^2\bar{\psi}\psi e^{b\varphi}+2\pi\mu^2 b^2 e^{2b\varphi}\,, \tag{0.25}$$

The energy-momentum tensor and the superconformal current are

$$T=-\frac{1}{2}(\partial\varphi\partial\varphi-Q\partial^2\varphi+\psi\partial\psi)\,, \tag{0.26}$$

$$G=i(\psi\partial\varphi-Q\partial\psi)\,. \tag{0.27}$$

The superconformal algebra is

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m^2-1)\delta_{m+n}\,, \tag{0.28}$$

$$[L_m, G_k] = \frac{m-2k}{2} G_{m+k}\,, \tag{0.29}$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}\,, \tag{0.30}$$

with the central charge

$$c_L = \frac{3}{2} + 3Q^2\,. \tag{0.31}$$

where

$$Q = b + \frac{1}{b}\,. \tag{0.32}$$

Here k and l take integer values for the Ramond algebra and half-integer values for the Neveu-Schwarz algebra.

Since in the Neveu-Schwarz sector of $N = 1$ SLFT we have besides Virasoro generators L_m, supercurrents generators G_k with half-generators k, descendant fields are broken into two representations of the Virasoro algebra of integer and half-integer level. Thus we have in Neveu-Schwarz sector two types of Virasoro primary fields. First type is associated with the vertex operators $N_\alpha = e^{\alpha\varphi}$, and have conformal dimensions

$$\Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha)\,. \tag{0.33}$$

The physical states have $\alpha = \frac{Q}{2} + iP$.
The second type is given by the supercurrent descendant:

$$\tilde{N}_\alpha = G_{-1/2}N_\alpha\,. \tag{0.34}$$

It has the conformal dimension

$$\Delta_{\tilde{\alpha}} = \frac{1}{2}\alpha(Q-\alpha) + \frac{1}{2}\,. \tag{0.35}$$

For brevity the N_α and $\tilde{N}_\alpha$ primary fields simply will be denoted by α and $\tilde{\alpha}$ correspondingly. Therefore, if for example we take in the relation (0.4) α, β_2 and β_5 of N_α type, it takes the form

$$S_{0\beta_2}\left[\int d\beta_4 F_{0,\beta_4}\left[\begin{array}{cc}\beta_2 & \alpha\\ \beta_2 & \alpha\end{array}\right] e^{2\pi i\Delta_{\beta_4}} F_{\beta_4,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \beta_2 & \beta_2\end{array}\right]\right.$$
$$\left.+\int d\beta_4 F_{0,\tilde{\beta}_4}\left[\begin{array}{cc}\beta_2 & \alpha\\ \beta_2 & \alpha\end{array}\right] e^{2\pi i\Delta_{\tilde{\beta}_4}} F_{\tilde{\beta}_4,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \beta_2 & \beta_2\end{array}\right]\right]$$
$$= e^{\pi i(2\Delta_\alpha+2\Delta_{\beta_2})} F_{0,\beta_5}\left[\begin{array}{cc}\alpha & \alpha\\ \alpha & \alpha\end{array}\right] S_{\alpha\beta_2}(\beta_5)\,.$$

Supersymmetric Racah-Wigner symbols for the supergroup $\mathrm{U}_q(\mathrm{osp}(1|2))$

$$S_b\left(\frac{y}{2}\right) S_b\left(\frac{y}{2}+\frac{Q}{2}\right) \equiv S_{\mathrm{NS}}(y) \equiv S_1(y),$$

$$S_b\left(\frac{y}{2}+\frac{b}{2}\right) S_b\left(\frac{y}{2}+\frac{b^{-1}}{2}\right) \equiv S_{\mathrm{R}}(y) \equiv S_0(y).$$

The subscript a of $S_a(y)$ is defined mod 2: $S_{a+2}(y) \equiv S_a(y)$.

$$\left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \overline{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_s \\ \alpha_t \end{smallmatrix}\right\}^{\nu_3\nu_4}_{\nu_1\nu_2} =$$

$$\frac{S_{\nu_4}(\alpha_s+\alpha_2-\alpha_1)S_{\nu_1}(\alpha_1+\alpha_t-\alpha_4)}{S_{\nu_2}(\alpha_t+\alpha_2-\alpha_3)S_{\nu_3}(\alpha_3+\alpha_s-\alpha_4)} I_{\alpha_s,\alpha_t}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array}\right]^{\nu_3\nu_4}_{\nu_1\nu_2},$$

where $\overline{\alpha}_4 = Q-\alpha_4$, $\nu_i = 0,1,\ i=1,2,3,4$, $\sum_{i=1}^4 \nu_i = 0 \bmod 2$, and

$$
I_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_{\nu_1\nu_2}^{\nu_3\nu_4} =
$$

$$
\begin{aligned}
& (-1)^{\nu_3\nu_2+\nu_4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sum_{\nu=0}^{1}(-1)^{\nu(\nu_2+\nu_4)}S_{1+\nu+\nu_3}(y+\gamma_1^{\circ})S_{1+\nu+\nu_4}(y+\gamma_2^{\circ})\\
\times\quad & S_{1+\nu+\nu_3}(y+\gamma_3^{\circ})S_{1+\nu+\nu_4}(y+\gamma_4^{\circ})S_{\nu}(-y+\beta_1^{\circ})S_{\nu+\nu_2+\nu_3}(-y+\beta_2^{\circ})\\
\times\quad & S_{\nu+\nu_2+\nu_3}(-y+\beta_3^{\circ})S_{\nu}(-y+\beta_4^{\circ})\frac{dy}{2\mathrm{i}}. \qquad (0.36)
\end{aligned}
$$

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]=\frac{\mathcal{N}_{NS}(\alpha_s,\alpha_2,\alpha_1)\mathcal{N}_{NS}(\alpha_4,\alpha_3,\alpha_s)}{\mathcal{N}_{NS}(\alpha_t,\alpha_3,\alpha_2)\mathcal{N}_{NS}(\alpha_4,\alpha_t,\alpha_1)}\left\{\begin{array}{cc|c}\alpha_1 & \alpha_3 & \alpha_s\\ \alpha_2 & \alpha_4 & \alpha_t\end{array}\right\}^{11}_{11}$$

$$F_{\alpha_s,\tilde{\alpha}_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]=\frac{\mathcal{N}_{NS}(\alpha_s,\alpha_2,\alpha_1)\mathcal{N}_{NS}(\alpha_4,\alpha_3,\alpha_s)}{\mathcal{N}_{R}(\alpha_t,\alpha_3,\alpha_2)\mathcal{N}_{R}(\alpha_4,\alpha_t,\alpha_1)}\left\{\begin{array}{cc|c}\alpha_1 & \alpha_3 & \alpha_s\\ \alpha_2 & \alpha_4 & \alpha_t\end{array}\right\}^{11}_{00}$$

$$F_{\tilde{\alpha}_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]=\frac{\mathcal{N}_{R}(\alpha_s,\alpha_2,\alpha_1)\mathcal{N}_{R}(\alpha_4,\alpha_3,\alpha_s)}{\mathcal{N}_{NS}(\alpha_t,\alpha_3,\alpha_2)\mathcal{N}_{NS}(\alpha_4,\alpha_t,\alpha_1)}\left\{\begin{array}{cc|c}\alpha_1 & \alpha_3 & \alpha_s\\ \alpha_2 & \alpha_4 & \alpha_t\end{array}\right\}^{00}_{11}$$

$$F_{\tilde{\alpha}_s,\tilde{\alpha}_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]=\frac{\mathcal{N}_{R}(\alpha_s,\alpha_2,\alpha_1)\mathcal{N}_{R}(\alpha_4,\alpha_3,\alpha_s)}{\mathcal{N}_{R}(\alpha_t,\alpha_3,\alpha_2)\mathcal{N}_{R}(\alpha_4,\alpha_t,\alpha_1)}\left\{\begin{array}{cc|c}\alpha_1 & \alpha_3 & \alpha_s\\ \alpha_2 & \alpha_4 & \alpha_t\end{array}\right\}^{00}_{00}$$

$$\Lambda(y;m;\omega) = \gamma^{(2)}\left(\frac{y+m\omega_1}{r};\omega_1,\frac{\omega_1+\omega_2}{r}\right)$$
$$\times\gamma^{(2)}\left(\frac{y+(r-m)\omega_2}{r};\omega_2,\frac{\omega_1+\omega_2}{r}\right).$$

For the fundamental region $0 \leq m \leq r$ one has

$$\Lambda(y,m,\omega) = \prod_{k=0}^{m-1}\gamma^{(2)}\left(\frac{y}{r}+\omega_2\left(1-\frac{m}{r}\right)+(\omega_1+\omega_2)\frac{k}{r};\omega\right)$$
$$\times\prod_{k=0}^{r-m-1}\gamma^{(2)}\left(\frac{y}{r}+\frac{m}{r}\omega_1+(\omega_1+\omega_2)\frac{k}{r};\omega\right), \qquad (0.37)$$

$$\Lambda(y,m+kr;\omega) = (-1)^{mk+r\frac{k(k-1)}{2}}\Lambda(y,m;\omega). \qquad (0.38)$$

$$\lim_{y\to\infty}\Lambda(y;m;\omega) = e^{-\frac{i\pi}{2}\left(\frac{1}{r}B_{2,2}(y;\omega)+\frac{m^2}{r}-m+\frac{r}{6}-\frac{1}{6r}\right)},$$

$$\text{for } \arg\omega_1 < \arg y < \arg\omega_2 + \pi,$$

$$\lim_{y\to\infty}\Lambda(y;m;\omega) = e^{\frac{i\pi}{2}\left(\frac{1}{r}B_{2,2}(y;\omega)+\frac{m^2}{r}-m+\frac{r}{6}-\frac{1}{6r}\right)},$$

$$\text{for } \arg\omega_1 - \pi < \arg y < \arg\omega_2.$$

Recall the relation between $S_{\rm NS}(x)$, $S_{\rm R}(x)$ and the function $\Lambda(y,m;\omega)$ for $r=2$. Setting $\omega_2=b$ and $\omega_1=b^{-1}$, $Q=b+b^{-1}$, and using the notation accepted in conformal field theory literature $\gamma^{(2)}(z;b,1/b)=:S_b(z)$, we obtain

$$\Lambda(y,0;b^{-1},b)=S_b\left(\frac{y}{2}\right)S_b\left(\frac{y}{2}+\frac{Q}{2}\right)\equiv S_{\rm NS}(y)\equiv S_1(y), \tag{0.39}$$

$$\Lambda(y,1;b^{-1},b)=S_b\left(\frac{y}{2}+\frac{b}{2}\right)S_b\left(\frac{y}{2}+\frac{b^{-1}}{2}\right)\equiv S_{\rm R}(y)\equiv S_0(y). \tag{0.40}$$

The subscript a of $S_a(y)$ is defined mod 2: $S_{a+2}(y)\equiv S_a(y)$. Recalling formula (0.38), we see that for $r=2$ it takes a simple form: $\Lambda(y,m+2k;\omega)=(-1)^{mk}\Lambda(y,m;\omega)$. This implies

$$\Lambda(y,2m;b^{-1},b)=S_1(y),\quad \Lambda(y,2m+1;b^{-1},b)=(-1)^m S_0(y)\,. \tag{0.41}$$

$$\int_{-i\infty}^{i\infty}\sum_{m=0}^{r-1}\frac{\prod_{a=1}^{6}\Lambda(y+s_a,n_a+m)\Lambda(-y+s_a,n_a-m)}{\Lambda(2y,2m)\Lambda(-2y,-(2m))}\frac{dy}{i}$$
$$=2r(-1)^{\epsilon}\prod_{1\leq a<b\leq 6}\Lambda(s_a+s_b,n_a+n_b).$$

where

$$\sum_{a=1}^{6} n_a=0\,,\quad \sum_{a=1}^{6} s_a=Q \qquad (0.42)$$

Define the function:

$$W_\epsilon(\underline{s},\underline{n};\omega) = \int_{-i\infty}^{i\infty} \sum_{m\in\mathbb{Z}_r+\epsilon} \frac{\prod_{a=1}^{8}\Lambda(s_a\pm y; n_a\pm m;\omega)}{\Lambda(\pm 2y;\pm 2m;\omega)} \frac{dy}{2ir\sqrt{\omega_1\omega_2}}$$

where $\Lambda(x\pm y; n\pm m;\omega) = \Lambda(x+y; n+m;\omega)\Lambda(x-y; n-m;\omega)$ and $n_a \in \mathbb{Z}+\epsilon\,, \epsilon = 0, \frac{1}{2}$. Also the following balancing constraints on the parameters s_j and n_j hold true:

$$\sum_{j=1}^{8} s_j = 2Q, \qquad \sum_{j=1}^{8} n_j = 0.$$

$$W_\epsilon(\underline{s},\underline{n};\omega) = W_\delta(\underline{\tilde{s}},\underline{\tilde{n}};\omega) \tag{0.43}$$
$$\times \prod_{1\le j<k\le 4} \Lambda(s_j+s_k;n_j+n_k;\omega) \prod_{5\le j<k\le 8} \Lambda(s_j+s_k;n_j+n_k;\omega)\,,$$
$$\tilde{s}_j = s_j+\xi, \quad \tilde{s}_{j+4} = s_{j+4}-\xi, \quad j=1,2,3,4, \tag{0.44}$$
$$\xi = \frac{1}{2}(\omega_1+\omega_2-\sum_{j=1}^{4} s_j)$$

$$\begin{cases} \tilde{n}_a = n_a - \frac{1}{2}(\sum_{b=1}^4 n_b), & a=1,2,3,4, \\ \tilde{n}_a = n_a + \frac{1}{2}(\sum_{b=1}^4 n_b), & a=5,6,7,8, \end{cases} \tag{0.45}$$

Here $\delta = 0, \frac{1}{2}$, and one should take $\delta = \epsilon$, if $\frac{1}{2}\left(\sum_{b=1}^4 n_b\right)$ is an integer, or otherwise $\delta \neq \epsilon$.

Collecting all we obtain:

$$S^{SL}_{\alpha\beta_2}(\beta_5) = \frac{\mathcal{N}_{NS}(\alpha,\beta_5,\alpha)}{\mathcal{N}_{NS}(\beta_2,\beta_5,\beta_2)} S_{\alpha\beta_2}(\beta_5) \tag{0.46}$$

$$S_{\alpha\beta_2}(\beta_5) = S_{0\beta_2}\frac{e^{-\frac{i\pi}{2}(\Delta_{\beta_5}-1/2)}}{S_1(\beta_5)}$$
$$\times\int_{-i\infty}^{i\infty}\sum_{m=0}^{1}\Lambda(y-\frac{Q}{2}+\beta_2+\frac{\beta_5}{2};m)\Lambda(y+\frac{Q}{2}-\beta_2+\frac{\beta_5}{2};m)$$
$$\times\Lambda(-y-\frac{Q}{2}+\beta_2+\frac{\beta_5}{2};-m)\Lambda(-y+\frac{Q}{2}-\beta_2+\frac{\beta_5}{2};-m)$$
$$\times e^{2i\pi y(-Q/2+\alpha)}dy\,.$$

$$S^{SL}_{\alpha\beta_2}(\tilde{\beta}_5) = \frac{\mathcal{N}_R(\alpha,\beta_5,\alpha)}{\mathcal{N}_R(\beta_2,\beta_5,\beta_2)} S_{\alpha\beta_2}(\tilde{\beta}_5) \qquad (0.47)$$

$$S_{\alpha\beta_2}(\tilde{\beta}_5) = S_{0\beta_2}\frac{e^{-\frac{i\pi}{2}(\Delta_{\beta_5}-1/2)}}{S_0(\beta_5)}$$

$$\times \int_{-i\infty}^{i\infty} \sum_{m=0}^{1} \Lambda(y-\frac{Q}{2}+\beta_2+\frac{\beta_5}{2};m-1)\Lambda(y+\frac{Q}{2}-\beta_2+\frac{\beta_5}{2};m+1)$$

$$\times\Lambda(-y-\frac{Q}{2}+\beta_2+\frac{\beta_5}{2};-m)\Lambda(-y+\frac{Q}{2}-\beta_2+\frac{\beta_5}{2};-m)$$

$$\times e^{2i\pi y(-Q/2+\alpha)}e^{i\pi m}dy$$

Conclusion

In this paper we calculated the matrix $S_{\alpha\beta_2}(\beta_5)$ in two cases. These results allow us to conjecture that in all cases it will be given by the function with various choice of discrete variables $k_a, t_a, \; a = 1, 2$ and N:

$$
\begin{aligned}
&F_{\alpha\beta_2}(\beta_5)(\underline{k}, \underline{t}) = \\
&\int_{-i\infty}^{i\infty} \sum_{m=0}^{1} \Lambda(y - Q/2 + \beta_2 + \beta_5/2; k_1 + m) \\
&\times \Lambda(y + Q/2 - \beta_2 + \beta_5/2; k_2 + m) \\
&\times \Lambda(-y - Q/2 + \beta_2 + \beta_5/2; t_1 - m) \\
&\times \Lambda(-y + Q/2 - \beta_2 + \beta_5/2; t_2 - m) e^{2i\pi y(-Q/2+\alpha)} e^{i\pi m N} dy
\end{aligned}
$$

This conjecture will be considered in future works.