Dynamical symmetry in generalized Calogero model

Tigran Hakobyan

Yerevan State University & Alinkanyan National Laboratory

RDP School & Workshop on Mathematical Physics,

August 19–24, 2023, Yerevan

Goals and methods

The goal of this talk is the study the dynamical symmetries of N-particle Calogero model with particles exchanges = generalized Calogero model.

The method uses a close analogy with isotropic harmonic oscillator in $d = N$ dimension.

In particular:

- The dynamical symmetry group of the $d = N$ isotropic oscillator is extended for the Calogero model with particle exchanges. The method uses nonlocal covariant, or Dunkl derivative, the related deformation of Weyl algebra, and Dunkl deformation of symplectic algebra.
- Dunkl-deformed $sp(2N)$ algebra contains $sl(2, R) \equiv so(1, 2)$ conformal subalgebra. The Calogero eigenstates (in spherical coordinates) are classified according to the conformal algebra lowest-weigh representations.
- The conformal structure of the integrals of motion of the unbound Calogero model with particle exchanges has been studied also. Any Liouville integral generate a tower of descendants which form a highest-state $sl(2, R)$ multiplet.

Dynamical symmetry of isotropic oscillator

The dynamical symmetries of the $d = N$ isotropic oscillator

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + \omega^2 x_i^2\right) = \sum_{i=1}^{N} a_i^+ a_i + \frac{N\hbar\omega}{2}$$

form semi-direct product of noncompact real symplectic group and Weyl group [Haskell & Wybourne, 1976]:

$$SP(2N) \ltimes W_N$$

- The W_N is generated by the unity and $a_i^{\pm}$.
- The $SP(2N)$ is generated by:
  1. The $U(N)$ symmetry $= \begin{cases} so(N) \text{ angular momentum: } \imath(a_i^+ a_j - a_j^+ a_i), \\ \text{Fradkin tensor: } a_i^+ a_j + a_j^+ a_i. \end{cases}$
  2. The step operators: $a_i^+ a_j^+, \quad a_i a_j$
- It contains the conformal subgroup $SP(2) \equiv SL(2, R)$ generated by Hamiltonian and

$$\sum_i (a_i^{\pm})^2.$$

Calogero model

The Calogero model describes $1d$ particles with $1/r^2$ interaction bound by harmonic potential [Calogero (1969,1971)],

$$H_{\mathrm{C}} = \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega^2 x_i^2) + \sum_{i<j}\frac{g(g\mp\hbar)}{(x_i - x_j)^2}$$

Most its properties, like (super)integrability, spectrum, wave functions, and conservation laws, are conditioned by modified version with particle exchanges (a generalized Calogero model) [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)]:

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + \omega^2 x_i^2\right) + \sum_{i<j}\frac{g(g-\hbar s_{ij})}{(x_i - x_j)^2}:$$

- s_{ij} permutes two particles: $x_i \leftrightarrow x_j$
- On *bosonic/fermionic* states $s_{ij} = \pm 1$ and H is reduced to H_{C}.

- Pass to a covariant momentum

$$p_i \longrightarrow \pi_i = -\imath\hbar\nabla_i$$

by deforming the derivative in momentum:

$$\partial_i \longrightarrow \nabla_i = \partial_i - \frac{g}{\hbar}\sum_{j\neq i}\frac{1}{x_i - x_j}s_{ij}.$$

- Operator ∇_i was constructed first by Dunkl [(1988)].

- Inverse-square Calogero interaction can be encapsulated into Dunkl momentum operator leading to a "covariant" oscillator [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)]:

$$H = \frac{\pi^2}{2} + \frac{\omega^2 x^2}{2}.$$

Properties of Dunkl operators

- Dunkl operators are nonlocal, flat, covariant derivatives:

$$[\pi_i, \pi_j] = 0.$$

- But the standard commutations with coordinates are changed:

$$[\pi_i, x_j] = -\imath S_{ij},$$

where

$$S_{ij} = (\delta_{ij} - 1) g s_{ij} + \delta_{ij}\left(\hbar + g\sum_{k\neq i} s_{ik}\right).$$

- In the $g = 0$ limit,

$$\pi_i = p_i, \qquad S_{ij} = \hbar\delta_{ij}$$

recovering the Heisenberg algebra commutations.

- This is a Dunkl analog of Weyl algebra, W_N, or Cherednik algebra.

- We set: $\hbar = \omega = 1$.

Spectrum generating operators

- Dunkl-operator analog of lowering-rising operators:

$$a_i^{\pm} = \frac{x_i \mp \imath \pi_i}{\sqrt{2}}.$$

- The commutators:

$$[a_i, a_j] = [a_i^+, a_j^+] = 0,$$
$$[a_i, a_j^+] = S_{ij}.$$

- The generalized Calogero Hamiltonian can be expressed in terms of them:

$$H = \frac{1}{2}\sum_i (a_i^+ a_i + a_i a_i^+)$$

- Operators $a_i^{\pm}$ obey a standard spectrum generating relations [Brink, Hansson, Vasiliev (1992); Minhatan, Polychronakos (1992)]:

$$[H, a_i^{\pm}] = \pm a_i^{\pm}.$$

Bosonic ground state wavefuntion must obey [Brink, Hansson, Vasiliev (1992)]

$$a_i\psi_0 = 0$$

The solution is:

$$\psi_0 = \prod_{i<j} |x_i - x_j|^g e^{-\frac{1}{2}\sum_i x_i^2}$$

with ground state energy

$$E_0 = \frac{N}{2} + g\frac{N(N-1)}{2}.$$

Bosonic excitations are generated by symmetrized creation operators:

$$|k_1, \dots, k_N\rangle = (A_1^+)^{k_1}(A_2^+)^{k_2}\dots(A_N^+)^{k_N}\psi_0,$$

$$A_l^+ = \sum_{i=1}^{N}(a_i^+)^l$$

with $k_i = 0, 1, 2\dots$ and energy

$$E_{k_1\dots k_N} = E_0 + k_1 + 2k_2 + \dots + Nk_N$$

Dynamical symmetry of Calogero model

The dynamical symmetries of the N-particle Calogero system with exchange terms,

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(\pi_i^2 + x_i^2\right) = \frac{1}{2}\sum_{i=1}^{N}\{a_i^+, a_i\}$$

is formed by the Dunkl-deformed symplectic and Weyl algebras.

- Deformed Weyl algebra is generated by $a_i^\pm$, and permutations s_{ij}
- Deformed $sp(2N)$ is formed by:
  1. The Dunkl-deformed unitary symmetry consisting of
     a) Dunkl $so(N)$ angular momentum
     b) Dunkl analog of Fradkin tensor
  2. The step operators: $a_i^+ a_j^+$ and $a_i a_j$

Dunkl deformation of symplectic algebra

- Bilinear combinations of lowering-rising operators form the Dunkl deformation of the symplectic algebra $sp(2N)$:

$$E_{ij} = a_i^+ a_j, \qquad F_{ij} = a_i a_j, \qquad F_{ij}^+ = a_i^+ a_j^+. \tag{1}$$

- They form a closed algebra with the deformed Weyl algebra. Nonzero commutators are (with Herm. conj.):

$$[E_{ij}, a_k^+] = a_i^+ S_{jk}, \qquad [F_{ij}, a_k^+] = S_{ik} a_j + a_i S_{jk}.$$

- The first set of generators, E_{ij}, form deformed $u(N)$ algebra. Together with permutations s_{ij}, they generate the symmetries of nonlocal Calogero model:

$$[H, E_{ij}] = 0, \qquad [H, S_{ij}] = 0.$$

- They obey a quadratic relation [Feigin, T.H. (2015); Correa, T.H., Lechtenfeld, Nersessian (2016)]:

$$E_{ij}(E_{kl} + S_{kl}) = E_{il}(E_{kj} + S_{kj}).$$

  It implies Dunkl deformation of $u(N)$ commutations:

$$[E_{ij}, E_{kl} + S_{kl}] = E_{il} S_{kj} - S_{il} E_{kj}.$$

Properties of the Dunkl unitary algebra

- The symmetry operators E_{ij} split into the antisymmetric and symmetric parts yielding Dunkl angular momentum and Fradkin tensors [Feigin (2003); Kuznetsov (1996); Feigin, T.H. (2015); Correa, T. H., Lechtenfeld, Nersessian (2016)]:

$$L_{ij} = \imath(E_{ij} - E_{ji}), \qquad I_{ij} = E_{ij} + E_{ji}.$$

- Casimir element of Dunkl angular momentum is [Feigin, T.H. (2015)]:

$$H_\Omega = L^2 + S(S - N + 2), \qquad S = \sum_{i<j} S_{ij}.$$

$$[L_{ij}, H_\Omega] = 0.$$

  H_Ω coincides with angular Hamiltonian, or spherical part of the Calogero model [Feigin, Lechtenfeld, Polychronakos (2013)].

- Dunkl angular momentum square may be expressed also as:

$$H_\Omega = -r^2\nabla^2 + r\partial_r(r\partial_r + N - 2) \tag{2}$$

where $r = \sqrt{x^2}$ is the radial coordinate.

Spectrum generating part of deformed $sp(2N)$

- Finally, remaining $N(N+1)$ operators, $F^{\pm}$, form Dunkl analog of staircase generators in symplectic group $SP(2N)$. They rise/lower the energy levels:

$$[H, F^{\pm}_{ij}] = \pm 2F^{\pm}_{ij}, \qquad F^{\pm}_{ij} = F^{\pm}_{ji}.$$

- Nontrivial commutators with $a^{\pm}_i$ are:

$$[F_{ij}, a^{+}_k] = S_{ik}a_j + a_i S_{jk}, \qquad [F^{+}_{ij}, a_k] = -S_{ik}a^{+}_j - a^{+}_i S_{jk}.$$

- The remaining (nontrivial) commutators are:

$$[F_{ij}, F^{+}_{kl}] = S_{ik}(E_{lj} + S_{lj}) + E_{ki}S_{lj} + a_i S_{jk} a^{+}_l + a^{+}_k S_{il} a_j,$$
$$[F_{ij}, E_{kl}] = S_{ik}F_{jl} + a_i S_{jk} a_l.$$

Deformed $sp(2N)$ is closed algebra

Commutators with $F^{\pm}$ in closed form.

- Commutations with Dunkl angular momentum:

$$[F^{\pm}_{ij}, L_{kl}] = \imath(S_{ik}F^{\pm}_{jl} + F^{\pm}_{il}S_{jk} - S_{il}F^{\pm}_{jk} - F^{\pm}_{ik}S_{jl}).$$

- For distinct values of indexes, the nontrivial commutators (with Herm. conj.):

$$[F_{ij}, F^{+}_{kl}] = E_{ki}S_{lj} + E_{kj}S_{li} + E_{li}S_{kj} + E_{lj}S_{ki},$$
$$[F_{ij}, E_{kl}] = F_{jl}S_{ik} + F_{il}S_{jk}.$$

- In case the indexes take only two distinct values:

$$[F_{ii}, F^{+}_{kk}] = (E_{ki} + E_{ik} + E_{kk} + E_{ii} + S_{ii})S_{ik} + g^2$$
$$[F_{ik}, F^{+}_{ik}] = (2E_{ii} + S_{ii})S_{ik} + S_{ii}(E_{kk} + S_{kk}) + E_{ii}S_{kk},$$

- In case of the four equal-valued indexes:

$$[F_{ii}, F^{+}_{ii}] = 4E_{ii} - \sum_{k\neq i}(E_{ii} + E_{kk} + E_{ik} + E_{ki})S_{ik} + S^2_{ii} - (N-1)g^2,$$

$$[F_{ii}, E_{ii}] = 2F_{ii} - \sum_{k\neq i}(F_{kk} + F_{ik})S_{ik}.$$

Conformal subalgebra

- The Dunkl deformed symplectic algebra contains a conventional (non-deformed) conformal subalgebra $sl(2,R) \equiv so(1,2)$ formed by

$$K_\pm = \frac{1}{2}\sum_i F^\pm_{ii}, \qquad K_3 = \frac{1}{2}H.$$

- Commutators:

$$[K_-, K_+] = 2K_3, \qquad [K_3, K_\pm] = \pm K_\pm.$$

- Commutators in invariant basis:

$$[K_\alpha, K_\beta] = -\epsilon_{\alpha\beta\gamma}K^\gamma, \qquad K_\pm = K_1 \pm K_2,$$

where index is risen by Minkowski metrics,

$$\gamma^{\alpha\beta} = \mathrm{diag}(1,-1,-1).$$

- The explicit form of both generators:

$$K_1 = \frac{1}{2}(r^2 - H), \qquad K_2 = -\frac{1}{2}r\partial_r - \frac{N}{4}.$$

Properties of conformal subalgebra

- Conformal algebra commutes with the Dunkl angular momentum,

$$[K_\alpha, L_{ij}] = 0.$$

- The Casimir element

$$K^2 = K^\alpha K_\alpha = K_1^2 - K_2^2 - K_3^2,$$

$$[K_\alpha, K^2] = 0$$

is expressed in terms of the Dunkl angular momentum square:

$$H_\Omega = -4K^2 - \left(\tfrac{1}{4}N - 1\right) N.$$

Equivalent form of Dunkl operator

- Under similarity transformation with respect to $\phi = \prod_{i<j} |x_i - x_j|^g$, the Dunkl operator maps to

$$\nabla'_i = \phi^{-1}\nabla_i\phi = \partial_i + \sum_{j\neq i} \frac{g}{x_i - x_j}(1 - s_{ij}).$$

An advantage of ∇'_i compared with ∇_i:

1. $\nabla'_i = \partial_i$ on a symmetric function.
2. ∇'_i satisfy the deformed analog of Leibniz rule:

$$\nabla'_i(\phi\psi) = \phi\nabla'_i\psi + \psi\nabla'_i\phi - g\sum_{k\neq i} \frac{(\phi - s_{ik}\phi)(\psi - s_{ik}\psi)}{x_i - x_k}$$

3. Reduces to standard Leibniz rule if at least one from $\phi(x)$ and $\psi(x)$ is symmetric.

Dunkl deformation of harmonic polynomials and Dunkl angular momentum square

- The deformed Laplace equation $\nabla'^2 h(x) = 0$ has polynomial solutions, which are Dunkl deformations of the usual harmonic polynomials [Dunkl, 1988]:

$$h_{\mathbf{n}}(x) = r^{2(E_0+m-1)} \nabla_1'^{n_1} \nabla_2'^{n_2} \dots \nabla_N'^{n_N} r^{2(1-E_0)}$$

  is homogeneous polynomial solution of degree $m = \sum_{i=1}^N n_i$.
- Symmetrization reduces $h_{\mathbf{n}}$ to symmetric deformed harmonic polynomial of degree $m = \sum_{l=1}^N l k_l$

$$h_{\mathbf{k}}^{\mathrm{sym}}(x) = r^{2(E_0+m-1)} \mathcal{D}_1^{k_1} \mathcal{D}_3^{k_3} \dots \mathcal{D}_N^{k_N} r^{2(1-E_0)},$$

$$\mathcal{D}_l = \sum_{i=1}^N \nabla_i'^l.$$

  Here k_2 is absent since $\mathcal{D}_2 r^{-2(E_0-1)} = 0$.
- Wavefunctions and spectrum of Dunkl angular momentum square can be obtained ($\alpha = E_0 + m - 1$):

$$H_\Omega\, \phi(x) h_{\mathbf{k}}^{\mathrm{sym}}(x) = \mathcal{E}_\alpha \phi(x) h_{\mathbf{k}}^{\mathrm{sym}}(x), \qquad \mathcal{E}_\alpha = \alpha^2 - (\tfrac{1}{2}N - 1)^2.$$

.

- In spherical coordinates, via Dunkl-deformed spherical harmonics, the eigenfunctions of generalised Calogero model are:

$$\Psi_{k_1 k_2 k_3 \ldots k_N}(x) = e^{-\frac{r^2}{2}} L^{\alpha}_{k_2}(r^2)\phi(x) h^{\mathrm{sym}}_{k_1 k_3 \ldots k_N}(x)$$

L^{α}_{n} = associated Laguerre polynomial.

- Conformal action on these wavefunctions varies only k_2:

$$\begin{aligned} K_- \Psi_{\ldots k_2 \ldots} &= -(\alpha + k_2)\Psi_{\ldots k_2 - 1 \ldots}, \\ K_+ \Psi_{\ldots k_2 - 1 \ldots} &= -k_2 \Psi_{\ldots k_2 \ldots}, \\ K_3 \Psi_{\ldots k_2 \ldots} &= \left[\tfrac{1}{2}(\alpha + 1) + k_2\right]\Psi_{\ldots k_2 \ldots}, \end{aligned}$$

with the $k_2 = 0$ lowest-state,

$$K_- \Psi_{k_1 0 k_3 \ldots} = 0, \qquad K_3 \Psi_{k_1 0 k_3 \ldots} = \tfrac{1}{2}(\alpha + 1)\Psi_{k_1 0 k_3 \ldots}.$$

Conformal spin of this multiplet:

$$s = \tfrac{1}{2}(\alpha + 1)$$

$\pi/2$ rotation in $sl(2, R)$

- A (nonunitary) $\varphi = \pi/2$ rotation in (K_2, K_3) plane of conformal algebra

$$K_\alpha \to K'_\alpha = \exp(\varphi K_1) K_\alpha \exp(-\varphi K_1).$$

exchanges the $\alpha = 1, 2$ directions:

$$K_2 \to K'_2 = -K_3, \qquad K_3 \to K'_3 = K_2,$$

- The rotated lowering/rising generators are:

$$K'_+ = -H_0 = \frac{1}{2}\nabla^2, \qquad K'_- = \frac{r^2}{2}.$$

Integrals of Calogero-Moser model and $sl(2,R)$ descendants

The Liouville integrals of motion of the unbound Calogero model H_0 are symmetric polynomials on Dunkl operators [Polykronakos, 1992]:

$$I_n = \sum_i \nabla_i^n, \qquad I_2 = -2H_0$$

- The adjoint action $\hat{X}f := [X, f]$ of the conformal generators

$$\hat{K}'_+ I_n = 0, \qquad \hat{K}'_3 I_n = \tfrac{1}{2} n I_n,$$

generates highest weight representation with conformal spin $s = \frac{1}{2}n$.

- The descendants with $l = 0, 1, \ldots, n$:

$$I_{n,l} = (\hat{K}'_-)^l I_n, \qquad \hat{K}'_3 I_{n,l} = (\tfrac{1}{2}n - l) I_{n,l}$$

- The rising generator acts on them in a standard way:

$$\hat{K}'_{+} I_{n,l} = \hat{H}_0 I_{n,l} = -l(n-l+1) I_{n,l-1}.$$

- The Casimir element takes the conversional value

$$\hat{K}^2 = -\tfrac{1}{4} n(n+1).$$

- The descendants of integral of motion are expressed via Weyl-ordered products of the coordinate and Dunkl operator:

$$I_{n,l} = (-1)^l \frac{n!}{(n-l)!} \sum_{i=1}^{N} \left(x_i^l \nabla_i^{n-l} \right)_W ,$$

which follows immediately from

$$[K'_{-}, \nabla_i] = -x_i, \qquad [K'_{-}, x_i] = 0.$$

Properties of Weyl ordered product

- Weyl order just symmetrizes over all possible products of two operators with given powers. Examples:

$$(ab)_W = \frac{1}{2}(ab+ba), \quad (a^2b)_W = \frac{1}{3}(a^2b+aba+ba^2),$$
$$(a^2b^2)_W = \frac{1}{6}(a^2b^2+abab+baba+ab^2a+ba^2b+b^2a^2).$$

- Weyl-ordered polynomials are obtained from the power expansion of noncommutative binomial:

$$(a+vb)^n = \sum_{l=0}^{n} \binom{n}{l} v^l \left(a^{n-l}b^l\right)_W .$$

- As a result, a generating polynomial, which gathers all descendants $I_{n,l}$, is a Newton's power sum:

$$I_n(v) := \sum_{l=0}^{n} (-v)^l I_{n,l} = \sum_{i=1}^{N} (\nabla_i - vx_i)^n .$$

- The unbound Calogero model is maximally superintegrable with $N-1$ additional integrals apart from Liouville integrals [Wojciekhowski, 1983; Kuznetsov, 1995].

- Additional integrals = highest states in product of two $sl(2,R)$ multiplets,

$$(n_1)\otimes(n_2)=(n_1+n_2)\oplus(n_1+n_2-2)\oplus\cdots\oplus(|n_1-n_2|)$$

  where $s=n/2$.

- The construction gas been applied without the Dunkl operators [T. H., Karakhanyan, Lechtenfeld, 2014].

Additional integrals via descendants

- Additional integrals may be constructed from descendants, $k \le |n_1 - n_2|$:

$$I^{n_1,n_2}_{n_1+n_2-k} = \sum_{l=0}^{k} \frac{(-1)^l}{2} \binom{n_1 - k + l}{l} \binom{n_2 - l}{k - l} \{I_{n_1,k-l}, I_{n_2,l}\}$$

- They are parameterized by conformal spin $n = 2s$, $n_i = 2s_i$:

$$\hat{K}_3' I^{2s_1,2s_2}_{2s} = s I^{2s_1,2s_2}_{2s}, \qquad s = s_1 + s_2 - k$$

Summary and further studies

Resuls in brief:

- The dynamical symmetry group of the $d = N$ isotropic oscillator is extended for the Calogero model with particle exchanges. In includes the Cherednik algebra and a Dunkl analog of the $sp(2N)$.
- The $sl(2, R)$ structure of the integrals of motion of the unbound Calogero model with particle exchanges has been studied using the representation theory. Any liouville integral generate a tower of descendants, which form a $sl(2, R)$ multiplet.
- The additional integrals, responsible for the superintegrability, are the highest states in the product of such multiplets.

Some problems:

- The construction of Poincare-Birkhoff-Witt (PBW) basis for the Dunkl $sp(2N)$ algebra as was done in case of $u(N)_g$ and $so(N)_g$.
- The dynamical symmetries of the Calogero-Coulomb model, i.e. a Dunkl version of the $so(1, N + 1)$ in pure Coulomb problem.
- The extension to Calogero models based on more general finite reflection groups.
- The $sl(2, R)$ conformal algebra exists in quasi-integrable systems with long-range interactions alike the truncated Calogero model.