Grassmannian sigma models as Gross-Neveu models

Dmitri Bykov

Steklov Mathematical Institute & ITMP (MSU)

Erevan, August 2023

Russian Science Foundation

Summary

[Anselm '1959
Gross-Neveu '1974
Witten '1978
Andrei-Lowenstein '1979
Destri-de Vega '1989]

- Class of integrable sigma models explicitly equivalent to Bose/Fermi Gross-Neveu models

[DB 2020+]

Related: [LeClair, Ludwig, '2000
Saleur, Schomerus -2010]

GN-model is the "φ^4-theory" of 2D!

Examples in this talk

- The $\mathbb{CP}^{n-1}$ model
- Grassmannian σ-models $U(n)$, $O(n)$, $Sp(2n)$
- SUSY $\mathbb{CP}^{n-1}$ models
- r-matrix deformed GN-models

SUSY deformed $\mathbb{CP}^1$-model ("super-sausage")

Main example: $\mathbb{CP}^{n-1}$ – model

$$S = \int d^2z \sum_{A,B} G_{A\bar{B}}\, \partial_\alpha U^A\, \partial_\alpha \overline{U^B} + \text{Top. term}$$

Fubini–Study metric

$$\text{GLSM:} \quad \mathbb{CP}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^* \mapsto \left\{ \sum_{A=1}^{n} |U^A|^2 = 1 \right\} / U(1)$$

$$S_{GLSM} = \int d^2z \sum_{A=1}^{n} D_\alpha U^A\, \overline{D_\alpha U^A}$$

$$\boxed{D_\alpha U^A = \partial_\alpha U^A - i A_\alpha U^A}$$

$\mathbb{CP}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^*$ "GIT-quotient"

[DB, 2020]

$U, V \in \mathbb{C}^n$ $\quad \Psi = \begin{pmatrix} U \\ \overline{V} \end{pmatrix}$ "Dirac boson"

$$\mathcal{L} = \overline{\Psi}_a \not{D} \Psi_a + \varkappa \left(\overline{\Psi}_a \frac{1+\sigma_3}{2} \Psi_a \right) \cdot \left(\overline{\Psi}_b \frac{1-\sigma_3}{2} \Psi_b \right)$$

U(1) ($\mathbb{C}^*$) covariant derivative

Chiral gauged GN-model

$$\mathcal{L} = V \bar{D} U + \overline{U} \cdot D \overline{V} + \varkappa (\overline{U} U)(\overline{V} V)$$

Eliminate $V, \overline{V}$ $\longmapsto$ $\mathcal{L} = \frac{1}{\varkappa} \frac{\overline{DU} \cdot \overline{D} U}{\overline{U} \cdot U}$ = $\mathbb{CP}^{n-1}$ model

Integrability and the PCM

Target space = group G

$g \in G$

Maurer-Cartan current: $j = -g^{-1}dg = i\left(j_z\, dz + \bar{j}_{\bar z}\, d\bar z\right)$

E.o.m. $dj - j \wedge j = 0$ (flatness)

$d{*}j = \bar\partial_{\bar z} j_z + \partial_z \bar{j}_{\bar z} = 0$ (conservation)

$\Leftrightarrow \bar\partial j_z + \frac{i}{2}[j_z, \bar{j}_{\bar z}] = 0$

Consider the special case

$j_z \in \mathfrak{g}_{\mathbb{C}}$

$j_z^N = 0$ for some N

Nilpotent orbit of $G_{\mathbb{C}}$

$\mathfrak{sl}_n$ minimal nilpotent orbit

Simplest case: $G = SU(n)$, $\dot{j}_{\bar z}^2 = 0$, $\mathrm{rk}(\dot{j}_{\bar z}) = 1$.

$$\dot{j}_{\bar z} = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

inv. under $U \mapsto \lambda U$, $V \mapsto \lambda^{-1} V$, $\lambda \in \mathbb{C}^*$

Generally $\dot{j}_{\bar z} = \underset{\text{column}}{U} \otimes \underset{\text{row}}{V}$ with $VU = 0$

Dimension $=$
$= n + n - 1 - 1 = 2(n-1) =$
$= 2 \dim(\mathbb{CP}^{n-1})$

$$\bar\partial \dot{j}_{\bar z} + \frac{i}{2}[\dot{j}_{\bar z}, \bar{\dot{j}}_{\bar z}] = 0 \implies \left.\begin{aligned} &\bar D U - \frac{i}{2}(\bar U U)\bar V = 0 \\ &\bar D V + \frac{i}{2}(V\bar V)\bar U = 0 \end{aligned}\right]$$

E.o.m. of the $\mathbb{CP}^{n-1}$ model

Other nilpotent orbits

Restrictions of principal chiral model to nilpotent orbit $\Rightarrow$ Generalized Gross-Neveu model

- Nilp. orbits classified by their Jordan forms
- Classical integrability of GN models found in [Neveu, Papanicolaou '1978]
- Relation to PCM (for fermionic models): [Zakharov, Mikhailov '1980]

Chiral and non-chiral GN models

$\psi \mapsto e^{i\alpha}\psi$, $\psi \mapsto e^{\beta \gamma_3}\psi$

chiral

General nilp. orbits $\mapsto$ non-chiral GN models
(unrelated to sigma models)

Example. sp_{2n} minimal orbit

$\mathbb{C}^{2n}/\mathbb{Z}_2$: take $\dot{j}_z \in sp_{2n}$, $\dot{j}_z^2 = 0$, $\mathrm{rk}\, \dot{j}_z = 1$.

Solution: $\dot{j}_z = W \otimes W^t \omega_{2n}$

$W \in \mathbb{C}^{2n}/\mathbb{Z}_2$

symplectic form on $\mathbb{C}^{2n}$ $(\omega_{2n}^t = -\omega_{2n})$

Chiral and non-chiral GN models

$$\bar\partial j_z + \frac{i}{2}[j_{\bar z}, j_z] = 0 \implies \bar\partial W + \frac{i}{2}(\bar W W) W^* = 0$$

Choose $\omega_{2n} = \left(\begin{array}{c|c} \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{array}\right)$, $W = \begin{pmatrix} U \\ V^t \end{pmatrix}$

Non-chiral GN-model

$$\mathcal{L} = \bar\psi^a \not\partial \psi^a + (\bar\psi^a \psi^a)^2 , \quad \text{where} \quad \psi = \begin{pmatrix} U \\ \bar V \end{pmatrix}$$

In general, one cannot eliminate U or V variables

$\Rightarrow$ Not a sigma model!

General orbits

Let $\mathcal{N}$ be a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$

$\Rightarrow$ $\overline{\mathcal{N}}$ = complex symplectic variety (singular)

Suppose $\{\mu : \mathbb{M} \mapsto \overline{\mathcal{N}}\}$ = (Springer) resolution of singularities [$\mathcal{N}$ = Richardson orbit]

Then $\mathbb{M} \simeq T^*\mathcal{F}$, μ = moment map = j_z [Fu, Namikawa '2006]

$\mathcal{F}$ = flag manifold (recall the dimensions!)

Conjecture: $\exists$ GN-formulation for σ-model with target space $\mathcal{F}$

SO- / Sp - Grassmannians

The conjecture has been verified for:

- all $SU(n)$ flag manifolds [DB, 2020]
- SO- and Sp-Grassmannians [DB & V. Krivorol, 2023]

Families of Grassmannians:

$$\frac{U(n)}{U(m)\times U(n-m)} \qquad \frac{O(n)}{U(m)\times O(n-2m)} \qquad \frac{Sp(2n)}{U(m)\times Sp(2n-2m)}$$

unitary

orthogonal

symplectic

m-planes $L\subset\mathbb{C}^n$

h_n - nondegenerate quadratic form on $\mathbb{C}^n$

$h_n(L,L)=0$

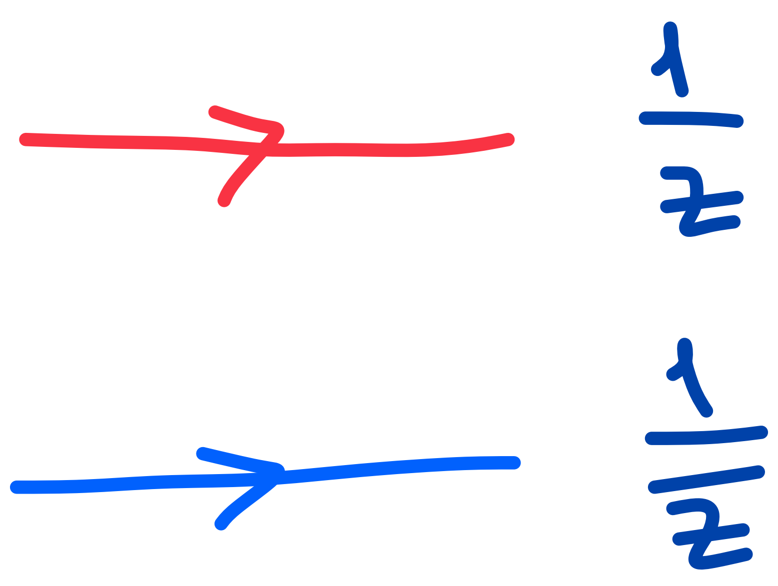
ω_{2n} - symplectic form on $\mathbb{C}^{2n}$

$\omega_{2n}(L,L)=0$

	Unitary	Orthogonal	Symplectic
Fields	$\mathbb{C}^m \underset{V}{\overset{U}{\rightleftarrows}} \mathbb{C}^n$	$\mathbb{C}^m \underset{V}{\overset{U}{\rightleftarrows}} \mathbb{C}^n$	$\mathbb{C}^m \underset{V}{\overset{U}{\rightleftarrows}} \mathbb{C}^{2n}$
Gauge constraints	$VU = 0$	$V h_n U = 0$ $U^t h_n U = 0$	$V \omega_{2n} U = 0$ $U^t \omega_{2n} U = 0$
Moment map $\mu = j_z =$	UV	$[UV - (UV)^t] h_n$	$[UV + (UV)^t] \omega_{2n}$
Nilpotency	$\mu^2 = U\underbrace{VU}V = 0$	$\mu^3 = 0$	$\mu^3 = 0$

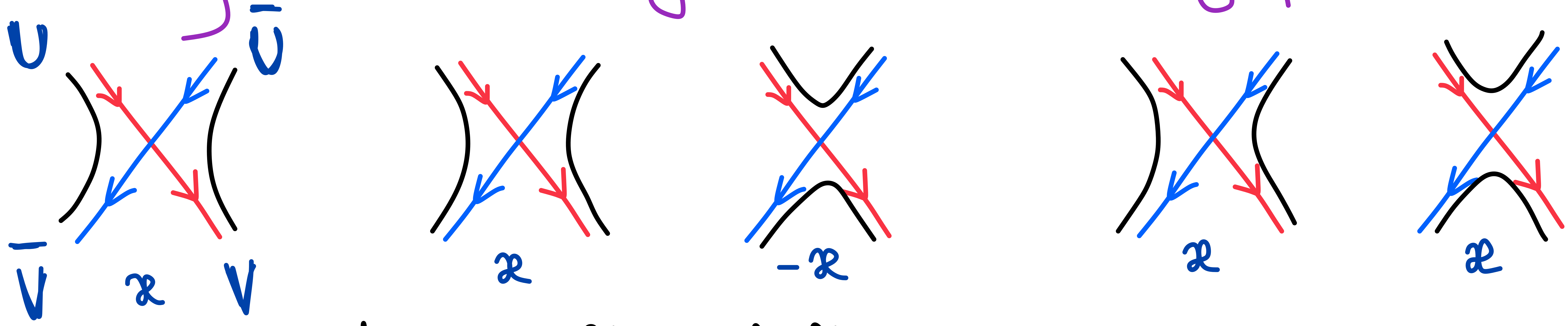
Feynman rules

Lagrangian: $\mathcal{L} = \mathcal{L}_{\text{free}} + \varkappa \, \mathrm{Tr}(\mu \bar{\mu})$



$\left(\mathbb{CP}^{n-1}: \quad \mu = U \otimes V \;\Rightarrow\; \mathrm{Tr}(\mu\bar{\mu}) = \sum_{i=1}^{n} |U_i|^2 \times \sum_{j=1}^{n} |V_j|^2 \right)$

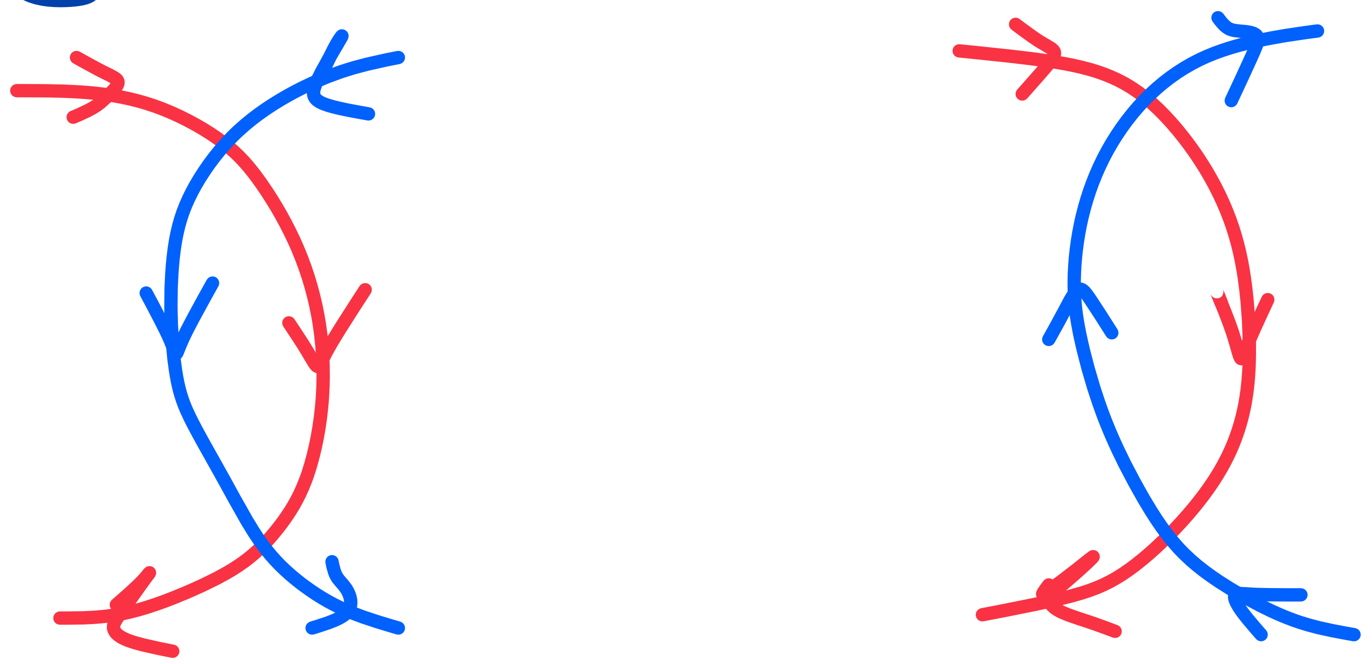
Unitary Orthogonal Symplectic



Black = flow of flavor

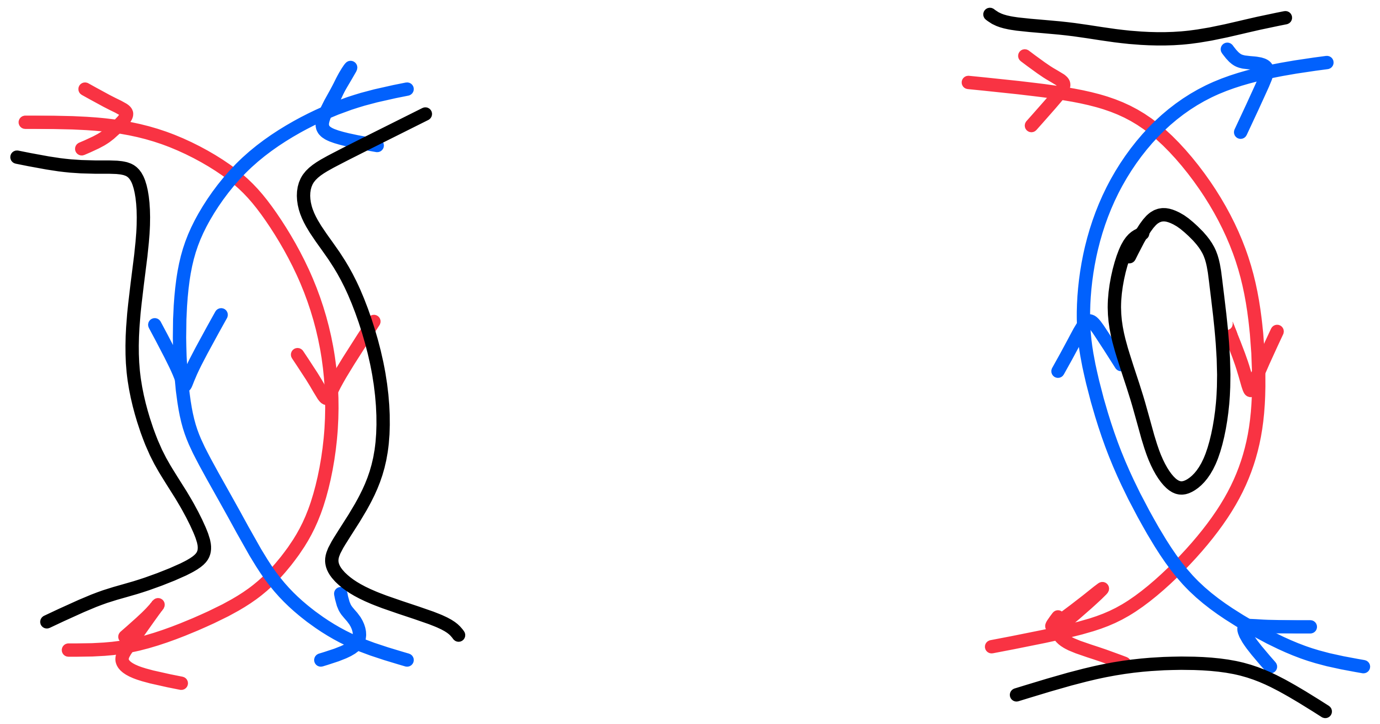
One-loop β-function

Divergent diagrams in all cases are of the form

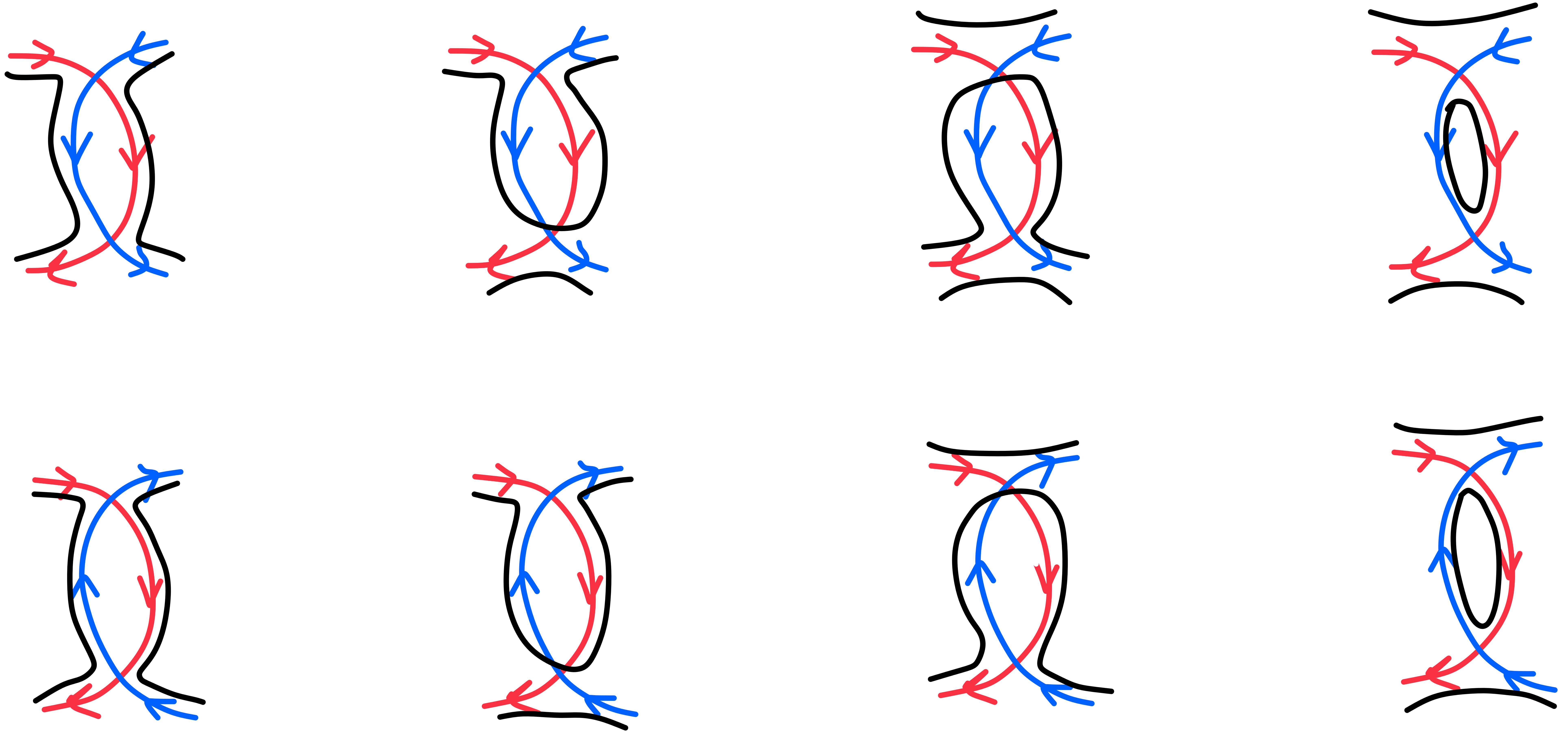


It remains to indicate flavor flow

Unitary



Orthogonal / Symplectic cases



O- and Sp-cases differ by signs

As a result, one gets

$$\beta_{U(n)} \sim n\, x^2$$

$$\beta_{O(n)} \sim (n-2)\, x^2$$

$$\beta_{Sp(2n)} \sim (n+1)\, x^2$$

$\sim h^\vee x^2$

(dual Coxeter number)

For symmetric spaces these are the correct (known) values!

$$\frac{U(n)}{U(m)\times U(n-m)} \qquad \frac{O(n)}{U(1)\times O(n-2)} \qquad \frac{O(2n)}{U(n)} \qquad \frac{Sp(2n)}{U(n)}$$

SUSY model

Main point: $\mathbb{CP}^{n-1}$ - model with worldsheet SUSY

= gauged version of a model with target space $GL(1|1)$ SUSY

Let $U, V, B, C \in \mathbb{C}^n$. Start with the free theory

$$\mathcal{L}_{\beta\gamma} := V \cdot \bar{\partial} U + \bar{U} \cdot \partial \bar{V} + B \cdot \bar{\partial} C + \bar{C} \cdot \partial \bar{B}$$

("$\beta\gamma$-system")

[Witten, Nekrasov '2005]

Bosons (commuting)

Fermions (anti-commuting)

(Super) symmetries

- TARGET-SPACE SUSY

One can rotate the doublets

$$\begin{pmatrix} U \\ C \end{pmatrix} \mapsto g \cdot \begin{pmatrix} U \\ C \end{pmatrix}, \quad (V \;\; B) \mapsto (V \;\; B) \cdot g^{-1} \qquad g \in GL(n|n)$$

- WORLDSHEET SUSY

[Friedan, Martinec, Shenker '1986]

$$\underbrace{\delta U = \varepsilon_1 C \qquad \delta B = -\varepsilon_1 V}_{Q_+} \qquad \underbrace{\delta C = -\varepsilon_2 \partial U \qquad \delta V = \varepsilon_2 \partial B}_{\overline{Q}_+}$$

[Kapustin '2005]

[Grassi, Policastro, Scheidegger '2007]

$$Q_+^2 = \overline{Q}_+^2 = 0 \qquad \{ Q_+, \overline{Q}_+ \} = \partial$$

Gauging

$$\mathcal{U} = \begin{pmatrix} U \\ C \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} V & B \end{pmatrix}$$

$$\bar{A}_{super} = \begin{pmatrix} \bar{A} & 0 \\ \bar{W} & \bar{A} \end{pmatrix}$$

Rewrite the Lagrangian:

$$\mathcal{L}_{CP} = \mathcal{V} \cdot \bar{\mathcal{D}}\, \mathcal{U} + \bar{\mathcal{U}} \cdot \mathcal{D}\, \bar{\mathcal{V}}, \qquad \bar{\mathcal{D}} = \bar{\partial} + i \bar{A}_{super}$$

Phase space: $T^*\mathbb{C}^{n|n} /\!/ G_\Delta$

$$G_\Delta := \left\{ g = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix} \in SL(1|1) \right\}$$

Bosonic $\mathbb{C}^*$ Fermionic $\mathbb{C}_F$

Supersymplectic reduction!

Interactions

Consider the Kac-Moody current

$$J = U\otimes V - C\otimes B \in gl_n$$

coupling constant $\left(\varkappa = \frac{1}{R^2}\right)$

and the Lagrangian $\boxed{\mathcal{L}_{int} = \mathcal{L}_{CP} + \varkappa\, \mathrm{Tr}(J\bar{J})}$

Level $k=0$ Kac-Moody current: $\langle J(z)\, J(w)\rangle = 0$

One gets the $\mathcal{N}=(2,2)$ SUSY σ-model!

Deformations

Vast literature on integrable deformations:
[Cherednik '1981] [Klimčík '2002]
[Fateev '1993+] [Delduc–Magro–Vicedo '2013]
[Hoare–Tseytlin] [Alfimov–Feigin–Litvinov]

Replace the GN Lagrangian:

$$\mathcal{L} = \bar{\Psi}_a \not{\partial} \Psi_a + (r_s)^{cd}_{ab} \left(\bar{\Psi}_a \frac{1+\gamma_3}{2} \Psi_c \right) \cdot \left(\bar{\Psi}_d \frac{1-\gamma_3}{2} \Psi_b \right)$$

classical r-matrix

s = deformation parameter

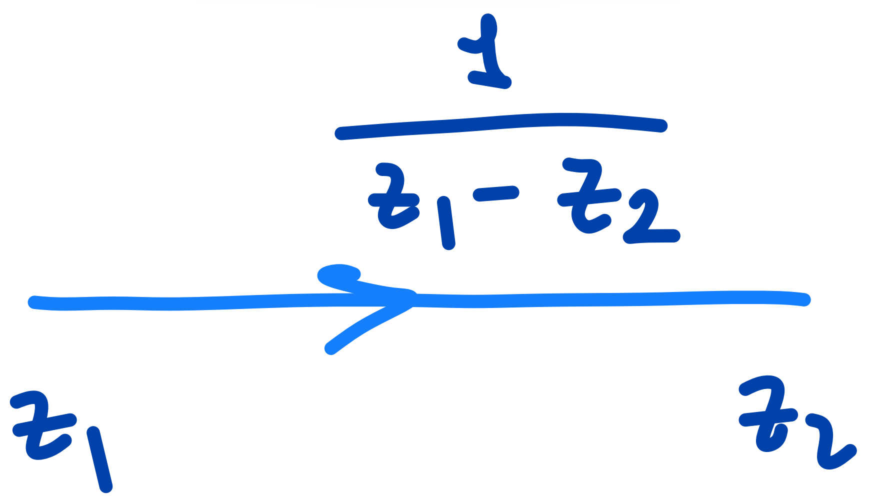
[Belavin Drinfeld '1980]

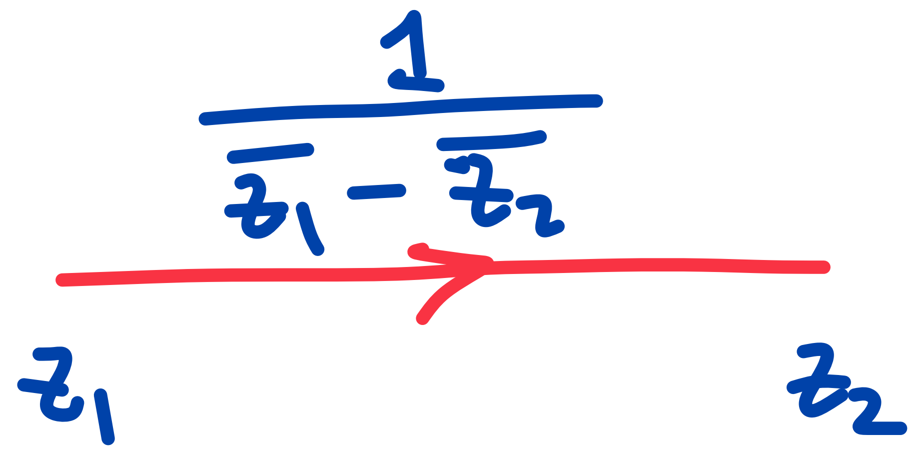
Noether current $= J\, dz + \bar{J}\, d\bar{z} \longmapsto \partial \bar{J} = [\bar{J}, r_s(J)]$

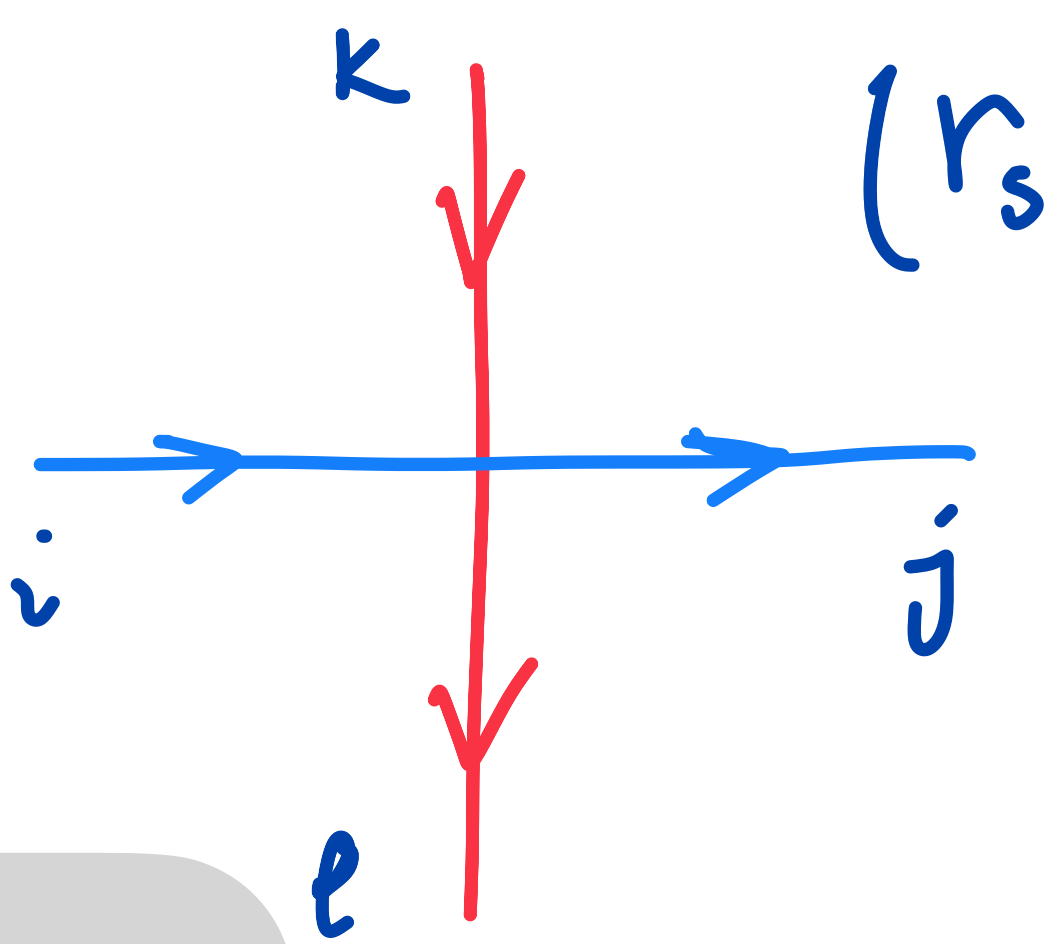
$$\boxed{\text{Flatness of } A_\beta := r_\beta(J)\, dz - r_{\beta s^{-1}}(\bar{J})\, d\bar{z}} \longleftrightarrow \boxed{\text{Classical Yang–Baxter equation}}$$

[Costello Yamazaki '2019]

One-loop β-function





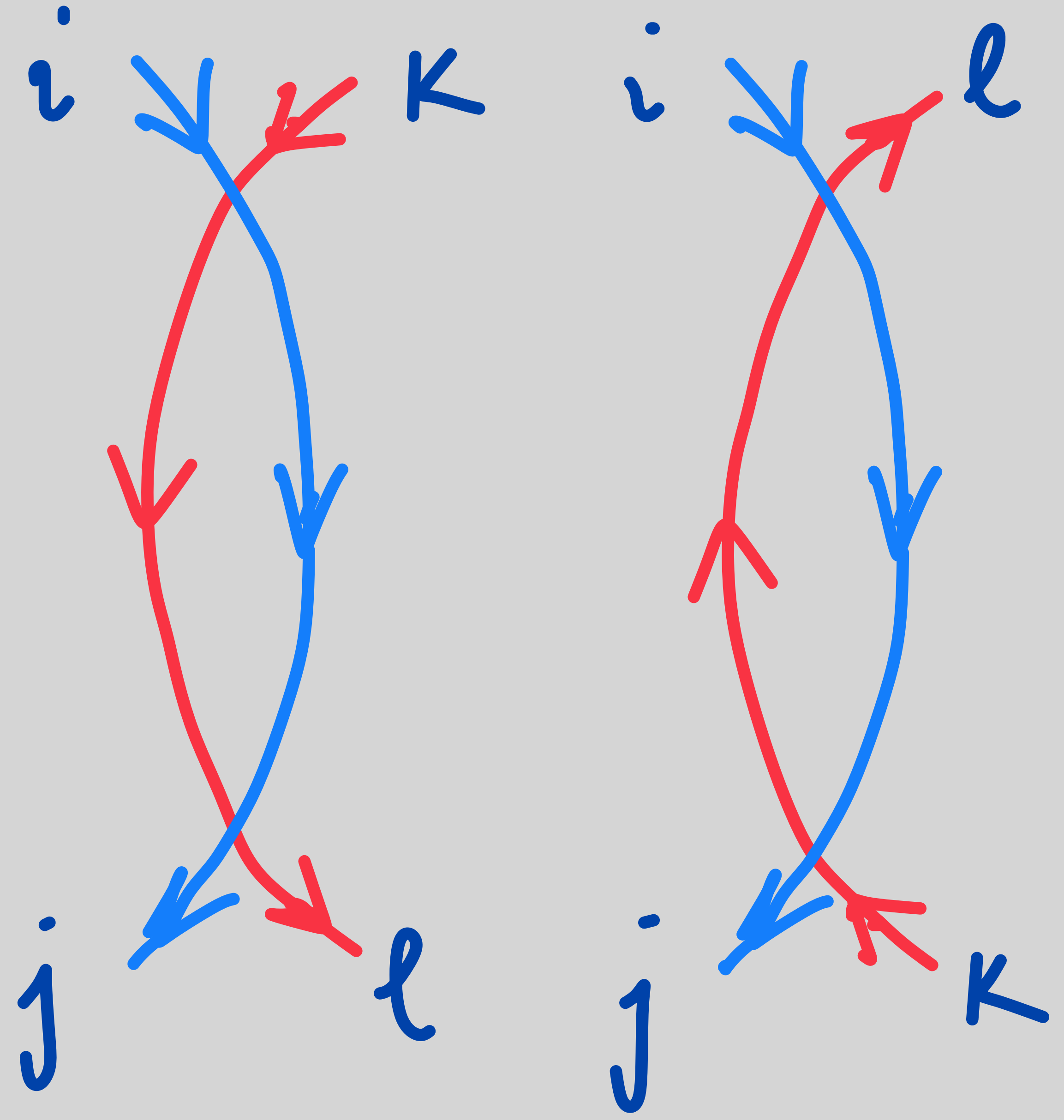


$$(r_s)^{k\ell}_{ij}$$

One-loop diagrams:

(No background field method necessary!)

$$\beta^{k\ell}_{ij} = \sum_{p,q=1}^{n} \left((r_s)^{kq}_{ip} (r_s)^{q\ell}_{pj} - (r_s)^{q\ell}_{ip} (r_s)^{kq}_{pj} \right)$$



RG-flow eqn. $\frac{d}{d\tau} r^{k\ell}_{ij} = \beta^{k\ell}_{ij} \longmapsto$

trigonometric deformation parameter

$\mathbb{CP}^{n-1}$

RG time

$$s = e^{n\tau}$$

The "sausage" model

$\mathbb{CP}^1$ $(n=2)$ $\Rightarrow$ $s = e^{2\tau}$

Deformed metric:

$$ds^2 = \frac{1}{\varkappa}\left(s^{-1/2} - s^{1/2}\right) \underbrace{\frac{|du|^2}{(s^{1/2} + |u|^2)(s^{-1/2} + |u|^2)}}_{= g_{u\bar{u}}\,|du|^2} \qquad 0<s<1$$

Ricci flow: $\quad -\dfrac{d}{d\tau} g_{v\bar{v}} = R_{v\bar{v}}$

[Fateev, Onofri, Zamolodchikov 1994]

$$\text{Length} \sim |\log(s)| \sim |\tau|$$

$s=0$ (UV limit: cylinder) $\mapsto$ $s=1$ (IR limit: $\mathbb{CP}^1$)

SUSY + deformation = SUSY sausage

$$\mathcal{L}_{def} = V \cdot \bar{D} U + \bar{U} \cdot D \bar{V} + \varkappa \, \mathrm{Tr}\left(r_s(J) \bar{J} \right)$$

$$\boxed{J = U \otimes V - C \otimes B}$$

- Choose the gauge $U_1 = 1$, $C_1 = 0$ ("inhomogeneous coordinates")
- Solve the constraints: $VU + BC = 0$, $BU = 0$

  $\mapsto$ one is left with u, v, b, c
- Integrate out $v, \bar{v}$:

Riemann tensor

$$\mathcal{L}_{def} = g_{u\bar{u}} |\bar{\partial} u|^2 + b \bar{D} c + \bar{c} D \bar{b} + R_{u\bar{u}u\bar{u}} \, g^{u\bar{u}} g^{u\bar{u}} \, b \bar{b} c \bar{c}$$

Standard $\mathcal{N} = (2,2)$ SUSY sigma model

The super-Thirring model

[joint with A. Pribytok, to appear]

UV limit $(s \mapsto 0)$

GN side

super-Thirring model

coupling $\varkappa$

$$\mathcal{L} = v\bar{\partial}u + \bar{u}\partial\bar{v} + b\bar{\partial}c + \bar{c}\partial\bar{b} + \\ + \varkappa\,|vu+bc|^2$$

[Freedman et. al. 1987+]

Sigma model side

super-cylinder $\quad ds^2 = \frac{1}{\varkappa}\frac{|du|^2}{|u|^2}$

Linear dilaton $\quad \Phi \sim \sqrt{\varkappa}\,\log|u|$

We will set $u = e^W$

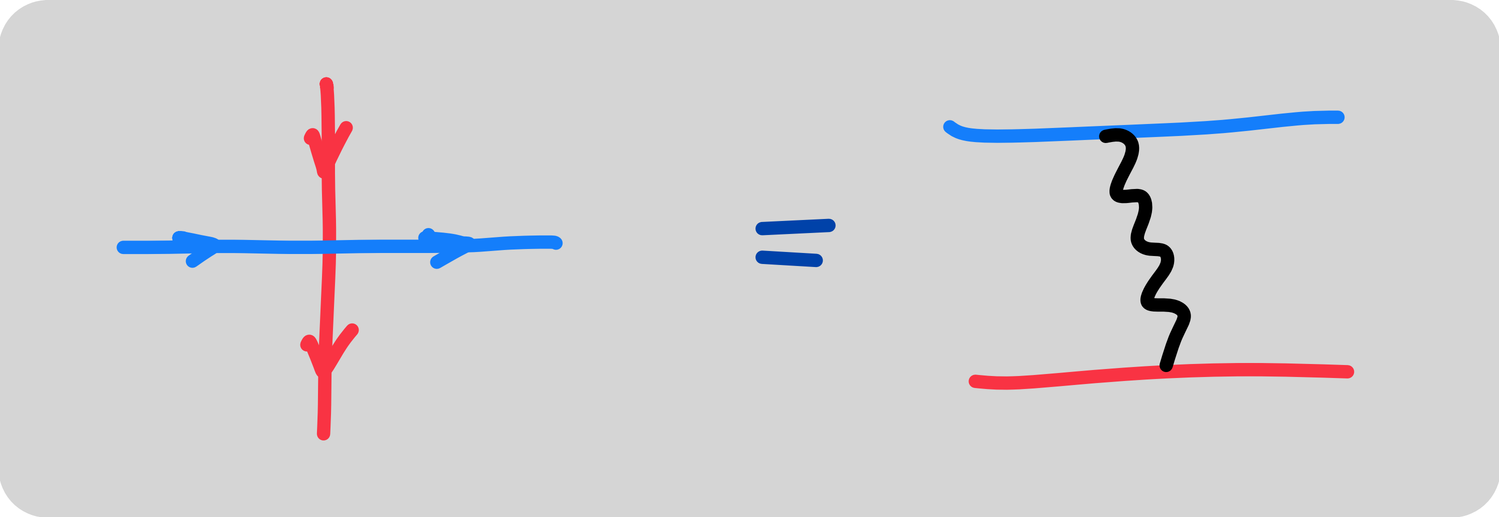
Equivalent!

Correlation functions (2pt)

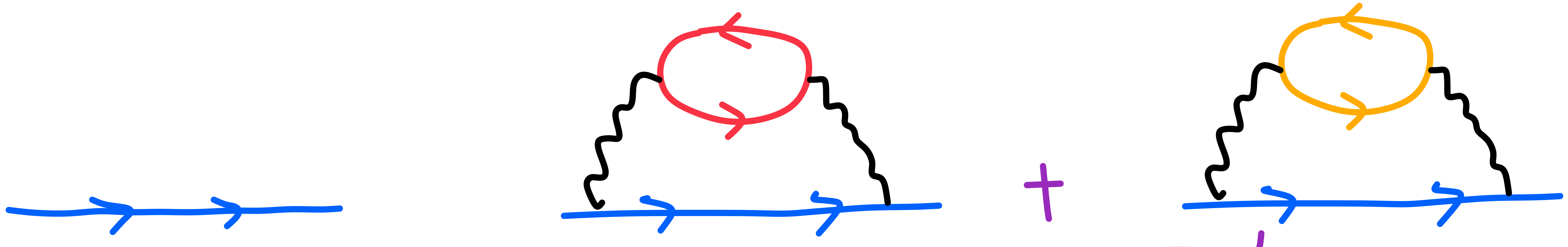
Integrating out $V, \bar{V}$, one sets $u = e^{W}$, $v = \frac{1}{x} e^{-W}\left(\partial \bar{W} - bc\right)$

2pt function: $\langle u(z_1)\, v(z_2) \rangle = \frac{1}{x} \int \partial \bar{W} \; e^{-\frac{1}{x}\int d^2 z\, |\bar{\partial} W|^2 + W(z_1) - W(z_2)} \dots$

Solving the e.o.m. $\bar{W} = x \ln\left(\frac{|z-z_1|^2}{|z-z_2|^2}\right)$, $W = 0$



$$\Rightarrow \langle u(z_1)\, v(z_2) \rangle = \frac{1}{z_1 - z_2}$$



= 0 (fermions cancel bosons)

Correlation functions (4pt)

Integrating over $V, \bar{V}$ one gets

$$\langle u(z_1)\, V(z_2)\, \bar{u}(\tilde{z}_1)\, \bar{V}(\tilde{z}_2)\rangle =$$

$$= -\frac{1}{\varkappa^2}\int \partial\bar{W}(z_2)\,\bar{\partial}W(\tilde{z}_2)\, e^{-\frac{1}{\varkappa}\int d^2z\, |\bar{\partial}W|^2 + W(z_1) + \bar{W}(\tilde{z}_1) - W(z_2) - \bar{W}(\tilde{z}_2)}$$

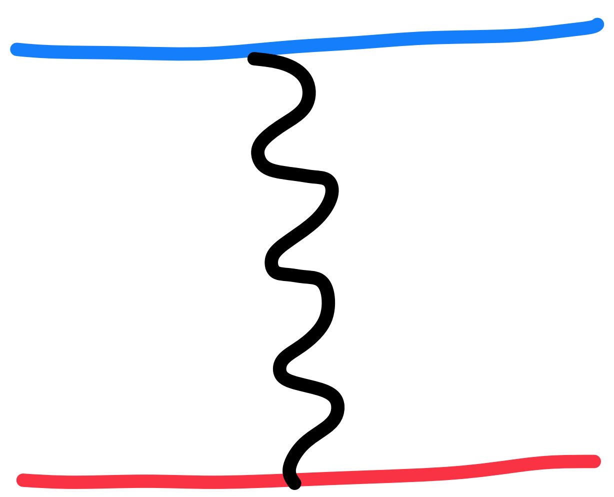
Solving the e.o.m.: $W = \varkappa \ln\left(\frac{|z-\tilde{z}_2|^2}{|z-\tilde{z}_1|^2}\right)$, $\bar{W} = \varkappa \ln\left(\frac{|z-z_2|^2}{|z-z_1|^2}\right)$

Thus,

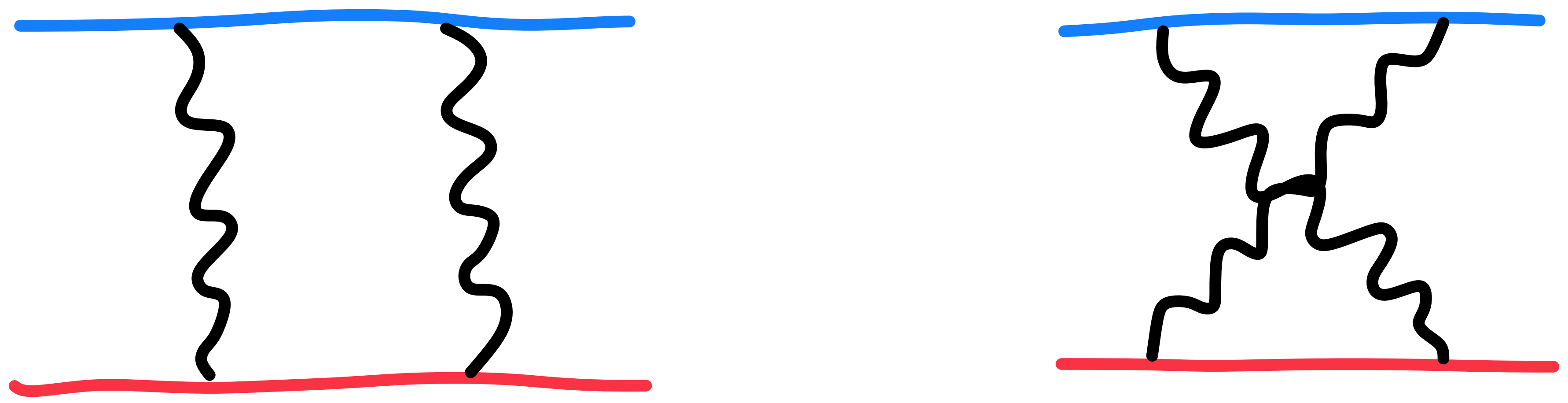
$$\langle \ldots \rangle = \frac{1}{(z_2-z_1)(\tilde{z}_2-\tilde{z}_1)}\left(\frac{|\tilde{z}_1-z_1|^2\,|\tilde{z}_2-z_2|^2}{|\tilde{z}_2-z_1|^2\,|\tilde{z}_1-z_2|^2}\right)^{\varkappa}$$

conformal cross-ratio

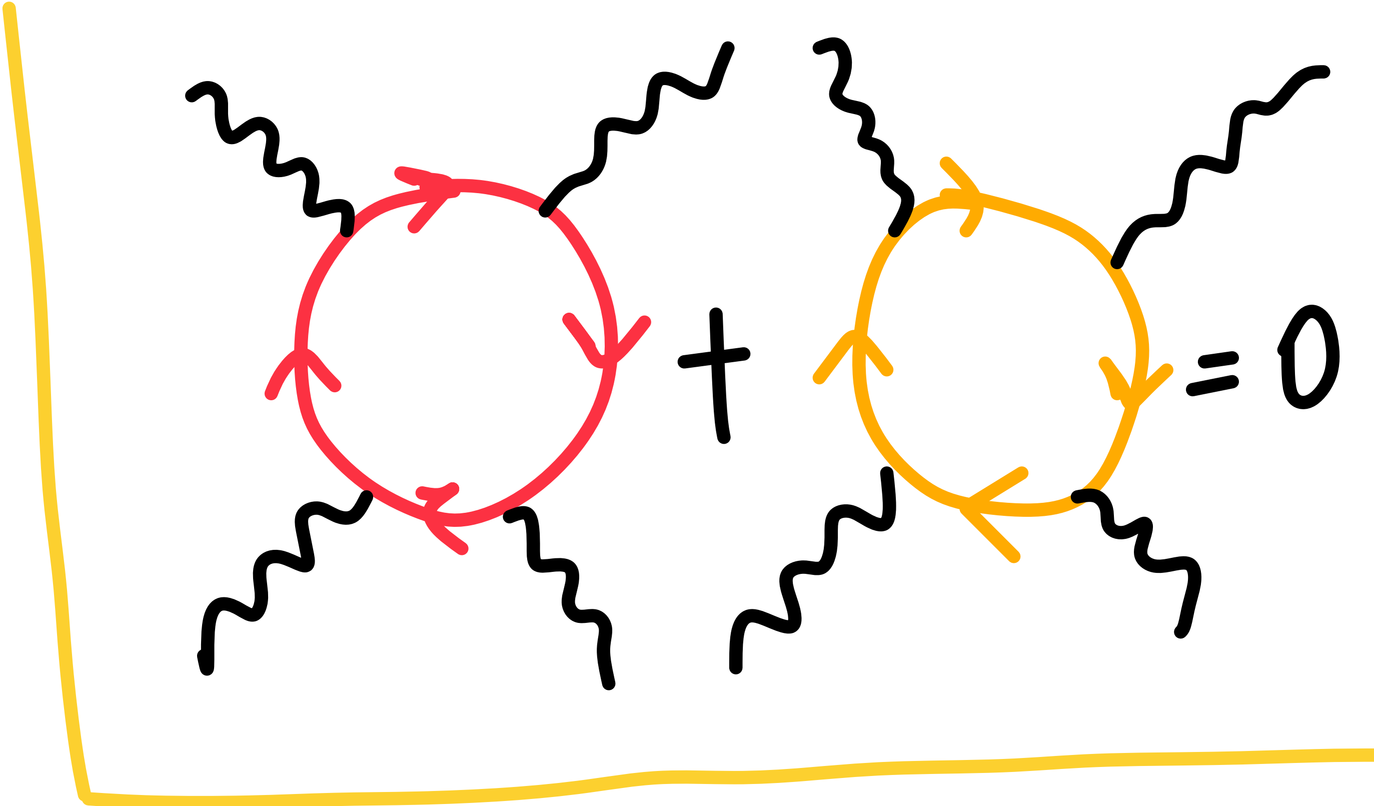
4-point function



Tree

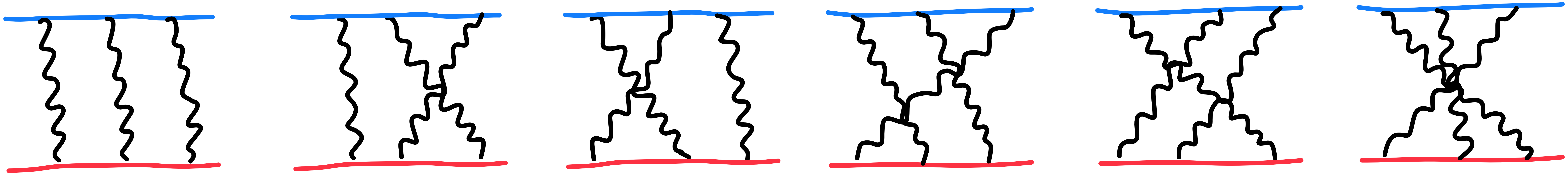


1-loop



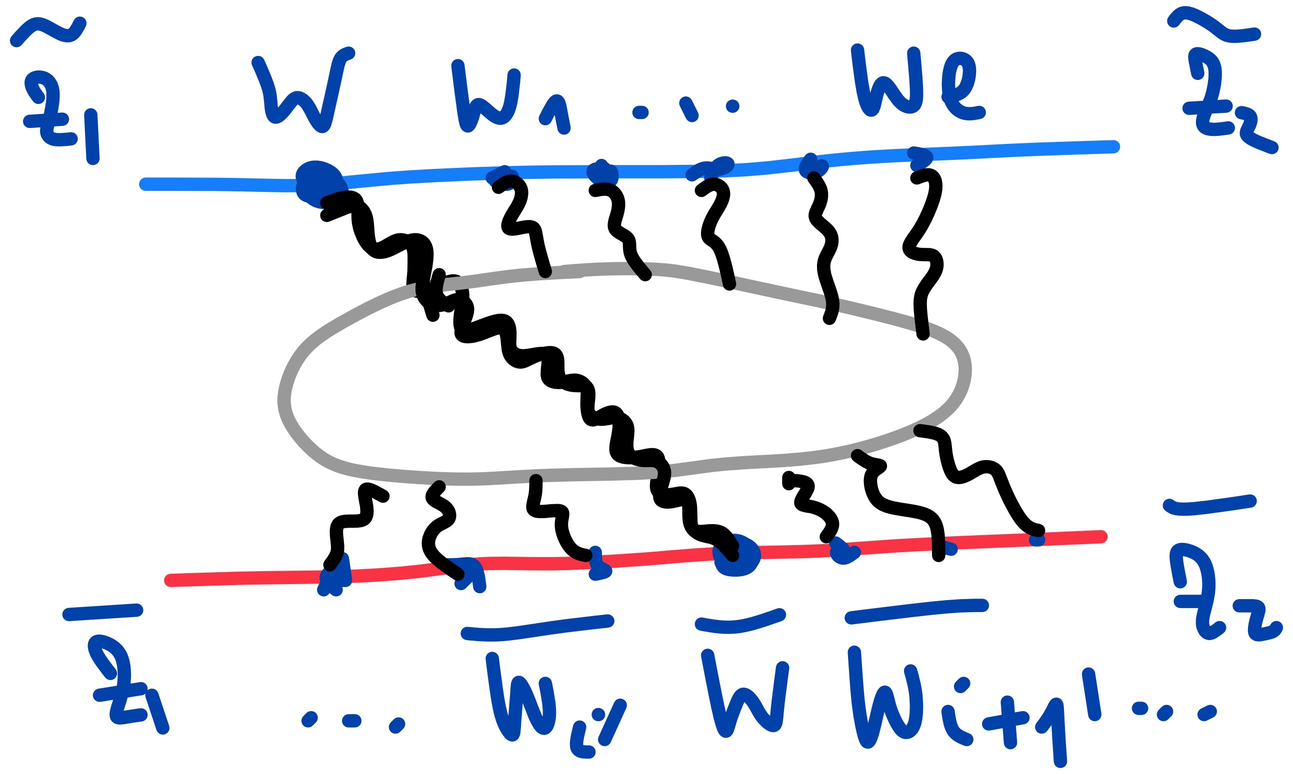
No loop insertions!

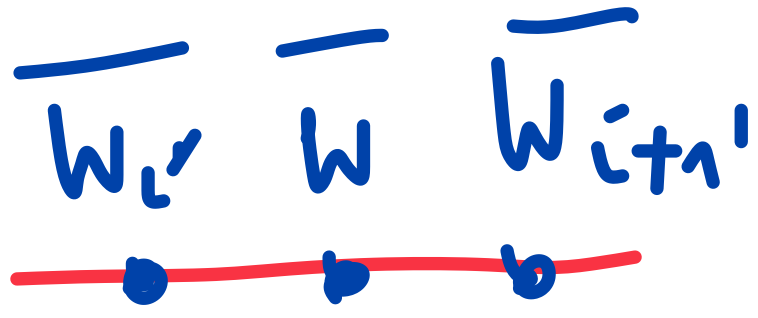
Only (crossed) ladder diagrams



2-loop (in general $(\ell+1)!$)

Recursion

At $\ell+1$ loops one has $\displaystyle\sum_{i=1}^{\ell}$ 

Use 

$$= \frac{1}{(\bar W_{i'}-\bar W)(\bar W-\bar W_{i+1'})} = \frac{1}{\bar W_{i'}-\bar W_{i+1'}}\left(\frac{1}{\bar W_{i'}-\bar W}+\frac{1}{\bar W-\bar W_{i+1'}}\right)$$

In the sum $\sum_{i=1}^{\ell}$ all but the first/last terms cancel out:

$$I_{\ell+1}(\tilde z_1,\tilde z_2|z_1,z_2) = x\int \frac{d^2W\,(\bar z_1-\bar z_2)}{(\tilde z_1-W)(\bar z_1-\bar W)(\bar W-\bar z_2)}\, I_\ell(W,\tilde z_2|z_1,z_2)$$

Solution: $I_\ell = \dfrac{1}{(\bar z_2-\bar z_1)(\tilde z_2-\tilde z_1)}\,\dfrac{x^{\ell+1}}{(\ell+1)!}\,(\text{cross-ratio})^{\ell+1}$

Q.E.D.

Summary & Outlook

- $\mathcal{E}$-models with flag manifold targets = GN-models
- SUSY generalizations & r-matrix deformations
- Proven quantum equivalence for super-Thirring / cylinder [DB, 2022]
- Exact β-function? [Morozov, Perelomov, Shifman '1984] [Gerganov, LeClair, Moriconi '2001] [Gamayun, Losev, Shifman '2023]
- Sigma models with other targets (spheres / AdS etc.)?