# Democratic Lagrangians and where to find them

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# References

**Based on:**

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# References

**See also:**

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# Motivation

**Some problems to solve during our lifetime**

- $S-$duality in gauge theory: Montonen-Olive duality and its various reincarnations/extensions. Can we make it manifest?
- Field-theoretical (classical) description of magnetic charges in the same footing as electric ones (local, Lorentz covariant?).
- Quantization of gauge theory. S-duality is the key?
- Non-abelian interactions of (chiral) $p-$forms. In particular, the $6d$ two-forms (related to $M5$ branes and $(2,0)$ theory).
- Electric-magnetic duality in gravity. Key to quantization?

# Duality symmetry of Maxwell equations

The most familiar example of duality symmetry – free Maxwell eq.'s:

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 0\,,$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 0\,,$$

invariant with respect to the duality rotations:

$$\overrightarrow{E} \to \cos\alpha\,\overrightarrow{E} + \sin\alpha\,\overrightarrow{B}\,,$$

$$\overrightarrow{B} \to -\sin\alpha\,\overrightarrow{E} + \cos\alpha\,\overrightarrow{B}\,.$$

Discrete duality – exchange of the electric $\overrightarrow{E}$ and magnetic $\overrightarrow{B}$ fields:

$$\overrightarrow{E} \to \overrightarrow{B}\,, \quad \overrightarrow{B} \to -\overrightarrow{E}\,.$$

# Duality symmetry of Maxwell equations

When the electromagnetic field is coupled to charged matter,

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j_e}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{E} = 4\pi \rho_e\,,$$

the duality symmetry is broken, unless one introduces magnetic charges – monopoles. These form a magnetic current $\overrightarrow{j_m}$:

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{j_m}\,, \quad \overrightarrow{\nabla} \cdot \overrightarrow{B} = 4\pi \rho_m\,.$$

The Maxwell equations remain duality invariant if the duality rotates also the four-vector currents $j_e^\mu = (\rho_e, \overrightarrow{j_e})$, $j_m^\mu = (\rho_m, \overrightarrow{j_m})$:

$$j_e^\mu \to \cos\alpha\, j_e^\mu + \sin\alpha\, j_m^\mu\,,$$

$$j_m^\mu \to -\sin\alpha\, j_e^\mu + \cos\alpha\, j_m^\mu\,.$$

# Duality symmetry of electromagnetic equations

Maxwell action (conventional) is not duality symmetric:

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x(\vec{E}^2 - \vec{B}^2)\,.$$

It changes the sign under discrete duality transformations.

Democracy requires employing two vector potentials: $A^1_\mu$ and $A^2_\mu$ with field strengths $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ $(a = 1, 2)$. Free Maxwell equations are equivalent to (twisted self-) duality relation:

$$\star F^a_{\mu\nu} = \epsilon^{ab}\, F^b_{\mu\nu}\,,$$

where

$$\star F^b_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho}F^{b\,\lambda\rho}\,, \quad \epsilon^{ab} = -\epsilon^{ba}\,, \quad \epsilon^{12} = 1$$

# Free democratic equations

Maxwell (Larmore) actions for $p-$forms and $(d-2-p)-$forms describe the same particle content.
When $d = 2p + 2$, the dual variables are of the same type and the Maxwell action itself takes the same form in both variables.

## Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$\star F = \pm G\,, \quad F = dA\,, \quad G = dB$$

## Duality-symmetric formulations

Zwanziger '70,..., Gaillard-Zumino '80, Bialynicki-Birula '83,..., Schwarz-Sen '93, Gibbons-Rasheed '95, Pasti-Sorokin-Tonin '96, Cederwall-Westerberg '97, Rocek-Tseytlin '99, Kuzenko-Theisen '00, Ivanov-Zupnik '02, Ivanov-Nurmagambetov-Zupnik '14,...

# Chiral $p-$forms in $d = 4k + 2$ Minkowski space

## Minkowski vs Euclidean

Since $\star^2 = (-1)^{\sigma+p+1}$ where $\sigma$ is the number of time directions, only even-forms can be self-dual (chiral) in Minkowski space.

## $p = 2k$ forms in $d = 4k + 2$ dimensions

For even $p-$form potentials in special dimensions the corresponding particles are not irreducible but contain two irreps — chiral and anti-chiral halves.

# Self-dual (Chiral) fields

There are special representations of the Poincaré algebra which are described by self-dual forms. The covariant equations describing such representations are given as:

$$\star F = \pm F\,, \quad F = dA$$

which implies the regular "Maxwell equations" $d \star F = 0$.

**Lagrangian?**

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91, ..., Pasti-Sorokin-Tonin '95,..., Sen '15,...

# Conventional non-linear electrodynamics

## Lagrangian for general non-linear electrodynamics (NED)

$$\mathcal{L} = \mathcal{L}(s,p)\,, \quad s = \frac{1}{2}\,F_{\mu\nu}\,F^{\mu\nu}\,, \quad p = \frac{1}{2}F_{\mu\nu}\,\star F^{\mu\nu}\,,$$

## Equations and duality transformations

$$d\,F = 0\,, \qquad d\,G = 0\,, \qquad G = \star\frac{\partial \mathcal{L}}{\partial F}\,,$$

Since now $G$ is non-linearly related to $F$, the duality rotations:

$$F \to \cos\alpha\,F + \sin\alpha\,G\,,$$

$$G \to -\sin\alpha\,F + \cos\alpha\,G\,,$$

are not automatically a symmetry of the theory.

# Duality symmetry in NED

## Duality-symmetry in conventional NED

$SO(2)$ duality symmetry implies (Gaillard, Zumino '80, Bialynicki-Birula '83, Gibbons, Rasheed '95):

$$F \wedge F = G \wedge G$$

that is satisfied for Lagrangians $\mathcal{L}(s, p)$ solving the equation:

$$\mathcal{L}_s^2 - \frac{2\,s}{p}\mathcal{L}_s\,\mathcal{L}_p - \mathcal{L}_p^2 = 1\,,$$

where $\mathcal{L}_s = \frac{\partial \mathcal{L}}{\partial s}$, $\mathcal{L}_p = \frac{\partial \mathcal{L}}{\partial p}$. There are a few solutions known, among them Maxwell and Born-Infeld. A few more solutions by M. Hatsuda, K. Kamimura and S. Sekiya '99. New solutions were found recently: M. Svazas '21 (master thesis), K.M. and M. Svazas '22.

# Manifest duality-symmetry?

**Different approaches**

- Zwanziger '70 (manifest duality-symmetry, non-manifest Lorentz)
- Henneaux-Teitelboim '88, Schwarz-Sen '93 (manifest duality-symmetry, non-manifest Lorentz)
- Pasti-Sorokin-Tonin '95 (manifest duality-symmetry and Lorentz, non-polynomial action for free theory, reproduces non-covariant approaches after gauge-fixing)

Manifest duality symmetry requires democracy.

# New action for Chiral fields

## Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,Q)^2 + \,a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

## Symmetries

$$\begin{aligned} \delta A = dU\,; \qquad\qquad & \delta R = dV\,; \\ \delta A = -\,a\,da \wedge W\,, \quad & \delta R = da \wedge W\,; \\ \delta A = -\,\frac{a\,\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,, \\ \delta a = \varphi\,, \quad \delta R = \frac{\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,. \end{aligned}$$

# Equations and symmetries

## Equations

$$E_a \equiv \frac{\delta \mathcal{L}}{\delta a} \equiv (F + a\,Q) \wedge \star Q + F \wedge Q = 0\,,$$
$$E_A \equiv \frac{\delta \mathcal{L}}{\delta A} \equiv d\,[\star(F + a\,Q)] + da \wedge Q = 0\,,$$
$$E_R \equiv \frac{\delta \mathcal{L}}{\delta R} \equiv d\,[a \star (F + a\,Q)] - da \wedge F = 0\,.$$

## Relations

$$E_R - a\,E_A = da \wedge [F + a\,Q - \star(F + a\,Q)] = 0$$

From here (for $(da)^2 \neq 0$):

$$F + a\,Q - \star(F + a\,Q) = 0$$

and $E_a \equiv [F + a\,Q - \star(F + a\,Q)] \wedge Q = 0$ follows from $E_A = 0 = E_R$.

# The spectrum

## Consequences of e.o.m.

From the equations of motion it follows that:

$$da \wedge dR = 0$$

which can be solved generally as:

$$R = d\lambda + da \wedge \rho$$

This implies that $R$ is pure gauge. In the $R = 0$ gauge, we get:

$$\star F = F$$

Thus the propagating d.o.f. consist of a single self-dual $p-$form.

# Democratic formulation for $p-$form in $d$ dimensions

## Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,P)^2 + \frac{1}{2}(G + a\,Q)^2 - \,a\,Q \wedge F + \,a\,G \wedge P$$

where $F = dA\,,\ G = dB\,,\ P = dS\,,\ Q = dR$. The fields $A$ and $S$ are $p-$forms, while $B$ and $R$ are $(d-p-2)-$forms.

This is a democratic formulation for a $p-$form field (together with dual $(d-p-2)-$form field) in $d$ dimensions. The equations imply that $S, R$ are pure gauge (as is the field $a$), and the only physical d.o.f. are in $A, B$, satisfying the duality relation:

$$\star\, dA + dB = 0\,.$$

# Nonlinear theories of chiral $p-$forms

## Free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q\,, \qquad (F = dA\,,\; Q = dR)\,.$$

## Self-interacting Lagrangian: general recipe

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q + g(H^-)\,,$$

$$H^- = F + a\,Q - \star(F + aQ)\,.$$

## Equations of motion

In the on-shell gauge $Q = 0$, the equations of motion are:

$$F + \star F = f(F - \star F)\,,$$

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,.$$

# Lorentz covariant equations

## A comment on nonlinear theory of chiral two-forms in 6d

"It appears that not only is there no manifestly Lorentz invariant action, but *even the field equation lacks manifest Lorentz invariance*."
Perry, Schwarz '96

## Equations of motion

$$F + \star F = f(F - \star F)\,.$$

This is the most general equation for non-linear self-dual $p-$form in $d = 2p + 2$ dimensions, with arbitrary $f : \Lambda^- \to \Lambda^+$.
Our Lagrangian formulation covers all such theories, for which

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,,$$

with some scalar $g(Y)$.

# Comparison to other formulations

| Formalism: | PST | Sen's | Clone field |
|---|---|---|---|
| Interactions by arbitrary functions | x | ✓ | ✓ |
| Auxiliary fields gauged away | ✓ | x | ✓ |
| Gauge potential as fundamental field | ✓ | x | ✓ |

More details in:
Oleg Evnin and K.M., "Three approaches to chiral form interactions", *Differ. Geom. and Appl.* 89 (2023), 102016 [arXiv:2207.00626].

# Derivation from Chern-Simons

## Main idea

Reduction proposed by Arvanitakis et al. in arXiv:2212.11412

## Chern-Simons with a boundary term

The action:

$$S_{\text{free}} = \int_M H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} H \wedge \star H$$

Full variation:

$$\delta S_{\text{free}} = 2\int_M \delta H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} \delta H^+ \wedge H^- \,.$$

$$H^{\pm} = H \pm \star H$$

# Reduction

## Main idea

Decompose the field as ($v = da$ satisfies $v^2 \neq 0$ on the boundary):

$$H = \hat{H} + v \wedge \check{H}\,, \qquad dv = 0\,.$$

Then the field $\check{H}$ is a Lagrange multiplier enforcing a constraint on the field $\hat{H}$,

$$v \wedge \mathrm{d}\hat{H} = 0\,,$$

with a solution (arXiv:2101.02350)

$$H = \mathrm{d}A + v \wedge R$$

Plugging this back into the action gives the chiral Lagrangian discussed earlier.

# Extensions to interacting chiral fields

## General equations

general equations describing self-interactions of a chiral field are given as

$$H^- = f(H^+)\,, \qquad \mathrm{d}H = 0\,,$$

where $f : \Lambda^+ \to \Lambda^-$ is an antiselfdual form valued function of a selfdual variable.

## Action

$$S = \int_M H \wedge \mathrm{d}H - \int_{\partial M} \frac{1}{2}\, H \wedge \star H + g(H^+)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$.

# Democratic $p-$forms

## Generalization to democratic case

Action:

$$S = \int_M (-1)^{d-p}\, G \wedge \mathrm{d}F + \mathrm{d}G \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) + g(F + \star G)\,,$$

with bulk equations $\mathrm{d}F = 0 = \mathrm{d}G$ and boundary equations:

$$F - \star G = f(F + \star G)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$ for a $(p+1)-$form argument $Y$.

## More

More details soon: Evnin, Joung, K.M., arXiv:2308.xxxxx.

# Type II Supergravities: preliminaries

## Some notations

Reflection operator: $\star\alpha = (-1)^{\lfloor \frac{\deg\alpha}{2} \rfloor + \deg\alpha} * \alpha\,,$

Mukai pairing: $(\alpha, \beta) := (-1)^{\lfloor \frac{\deg\alpha}{2} \rfloor} (\alpha \wedge \beta)^{top},$

Differential: $D\alpha = d\alpha + H \wedge \alpha\,.$

## Properties

$$(\alpha, \star\beta) = (\beta, \star\alpha)\,, \qquad \star^2 = 1\,, \qquad D^2 = 0\,,$$

$$\int_M (\alpha, D\beta) = -\int_M (D\alpha, \beta) \qquad \text{(up to boundary terms)}\,,$$

$$D(f\alpha) = f D\alpha + df \wedge \alpha\,, \qquad \text{(for any function } f\text{)}.$$

## Action for Type II SUGRAS

$$S = S_{NS} + S_{RR}$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int \left[ \sqrt{-g}\, e^{-2\varphi} \left( \mathcal{R} + 4(d\varphi)^2 - \frac{1}{12} H^2 \right) \right],$$

$$S_{RR} = \pm \frac{1}{8\kappa^2} \int \left[ \frac{1}{2}(F + aQ, \star(F + aQ)) + (F, aQ) \right],$$

$$F = DA, \quad Q = DR\,.$$

Upper/lower sign corresponds to IIA/IIB.

## Field content

$$F = F_2 + F_4 + F_6 + F_8 + F_{10}, \qquad \text{(IIA case)}$$

$$F = F_1 + F_3 + F_5 + F_7 + F_9. \qquad \text{(IIB case)}$$

# Democratic type II SUGRA

## On-shell reduction

On-shell one can gauge fix $Q = 0$,

$$DF = 0\,, \qquad \star F = F\,.$$

reproducing democratic equations for RR forms.

# Manifest $SL(2,R)$-symmetric type IIB SUGRA

## Action

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\left\{\mathcal{R} - 2[(d\phi)^2 + e^{2\phi}(d\ell)^2] - \frac{1}{3}e^{-\phi}H^2 \right.$$
$$\left. - \frac{1}{3}e^{\phi}(H' - \ell H)^2\right\} + S_{SD}\,,$$

where

$$S_{SD} = \frac{1}{16\kappa^2}\int [(F + aQ)\wedge *(F + aQ) + 2\,F\wedge aQ$$
$$-2\,(1 + *)(F + aQ)\wedge X + X\wedge *X]\,,$$

and

$$X = \frac{1}{2}(B\wedge H' - B'\wedge H)$$

# Outlook

**A list of related ambitious problems**

- Democratic formulation for non-abelian gauge theory.
- Interacting theory of non-abelian (chiral) $p-$forms.
- Extensions to gravity and beyond (higher spins on my mind).

Thank you!

# Duality-symmetric Electromagnetism

### The Lagrangian for a single massless spin-one field in $d = 4$

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} H^b_{\mu\nu} H^{b\mu\nu} + \frac{1}{4}\, \epsilon_{bc}\, \varepsilon^{\mu\nu\lambda\rho}\, a\, F^b_{\mu\nu}\, Q^c_{\lambda\rho}$$

where $H^b_{\mu\nu} \equiv F^b_{\mu\nu} + a\, Q^b_{\mu\nu}$, $b = 1, 2$, and

$$F^b_{\mu\nu} = \partial_\mu\, A^b_\nu - \partial_\nu\, A^b_\mu\,, \quad Q^b_{\mu\nu} = \partial_\mu\, R^b_\nu - \partial_\nu\, R^b_\mu\,.$$

This Lagrangian describes a single Maxwell field, using 4 vectors and 1 scalar. Any solution of the e.o.m. is gauge equivalent to that of

$$R^b_\mu = 0, \qquad \star F^a_{\mu\nu} + \epsilon^{ab}\, F^b_{\mu\nu} = 0\,,$$

with a single propagating Maxwell field.

# Non-linear electrodynamics: Ansatz

## Ansatz for the consistent non-linear Lagrangian

$$\mathcal{L} = a\,\epsilon_{bc} F^b \wedge Q^c + f(U^{ab}, V^{ab})$$

where

$$U^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu}\, H^{b\mu\nu}\,, \quad V^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu} \star H^{b\mu\nu}$$

All symmetries are built in, except for the shift of $a$. The latter will fix the form of $f(U, V)$.

## Equations of motion

$$E_{A^b} \equiv d[(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c + a\,\epsilon_{bc}\, Q^c] = 0\,,$$

$$E_{R^b} \equiv d[a\{(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c - \epsilon_{bc}\, F^c\}] = 0\,.$$

# Imposing the missing symmetry

## Shift symmetry $\delta a = \varphi$

Equations of motion for $a$:

$$E_a \equiv Q^b \wedge\, K_b = 0\,,$$

where

$$K_a \equiv (f^U_{ab} + f^U_{ba}) \star H^b - (f^V_{ab} + f^V_{ba})\, H^b - \epsilon_{ab}\, H^b\,,$$

and $f^U_{ab} \equiv \partial f / \partial U_{ab}$, $f^V_{ab} \equiv \partial f / \partial V_{ab}$ ($f^U_{21} \equiv 0 \equiv f^V_{21}$).

Note, that $E_{R^b} - a\, E_{A^b} = da \wedge K_b = 0$, which implies $K_b = 0$ iff

$$K_a \pm \epsilon_{ab} \star K_b \equiv 0$$

Then, the $E_a = 0$ is redundant, implying the shift symmetry for $a$.

# The general democratic non-linear electrodynamics

## Solution

The equation $K_a \pm \epsilon_{ab} \star K_b \equiv 0$ implies

$$\pm\, \delta^{ac}\, (f^U_{cb} + f^U_{bc}) - \epsilon^{ac}\, (f^V_{cb} + f^V_{bc}) + \delta^a_b = 0$$

The general solution gives the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\, G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\tfrac{1}{2}\, G_{\mu\nu}\, G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

## Reminder: non-linear electrodynamics in the conventional language

$$S = \int \mathcal{L}(s,p)\, d^4x, \quad s \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad p \equiv \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu}$$

# Duality symmetry

## Discreet duality symmetry

Under the discrete duality,

$$\lambda_1 \to -\lambda_1\,, \qquad \lambda_2 \to -\lambda_2$$

Theories with such symmetry will satisfy:

$$g(-\lambda_1, -\lambda_2) = g(\lambda_1, \lambda_2)$$

# Duality symmetry

## Continuous duality symmetry

Under continuous duality symmetry,

$$\lambda_1 \to \cos(2\alpha)\,\lambda_1 + \sin(2\alpha)\,\lambda_2\,,$$
$$\lambda_2 \to -\sin(2\alpha)\,\lambda_1 + \cos(2\alpha)\,\lambda_2$$

Theories with such symmetry will have:

$$g(\lambda_1, \lambda_2) = h(w)\,, \qquad w = \sqrt{\lambda_1^2 + \lambda_2^2}$$

The corresponding Lagrangian is given as:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,,$$

where $w$ can be also given as:

$$w = \sqrt{-\det \mathcal{H}}\,, \qquad \mathcal{H}^{ab} \equiv (\star H^a_{\mu\nu} - \epsilon^{ac} H^c_{\mu\nu})(\star H^{b\mu\nu} - \epsilon^{bd} H^{d\mu\nu})/2$$

# Conformal symmetry

## Requirement of conformal symmetry

Requirement of conformal invariance translates into:

$$\lambda_1 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_1} + \lambda_2 \frac{\partial g(\lambda_1, \lambda_2)}{\partial \lambda_2} = g(\lambda_1, \lambda_2)$$

which can be solved, e.g. as:

$$g = \lambda_1 \, \tilde{g}(\lambda_1/\lambda_2)$$

# Conformal and duality-symmetric theory

## Conformal symmetry for duality-symmetric theories

This case gives:

$$w\,\frac{\partial h(w)}{\partial w} = h(w)\,,$$

which is solved by a linear function:

$$h(w) = \delta\, w$$

General conformal and duality-symmetric electrodynamics is given by the one-parameter Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\,\epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

# Equations of motion

## Equations

E.o.m. imply in $R^a = 0$ gauge:

$$\star F^1 + F^2 = g_2 \left(\star F^1 - F^2\right) - g_1 \star \left(\star F^1 - F^2\right),$$

where $g_1 \equiv \partial g/\partial\lambda_1,\ g_2 \equiv \partial g/\partial\lambda_2$.
One can solve from here $F^1$ in terms of $F^2$:

$$F^1 = \alpha(s,p) F^2 + \beta(s,p) \star F^2\,,$$

where $s = \frac{1}{2} F^2_{\mu\nu} F^{2\mu\nu}\,,\ p = \frac{1}{2} F^2_{\mu\nu} \star F^{2\mu\nu}$. One can now make contact with the single-field formalism with Lagrangian $\mathcal{L}(s,p)$ via

$$\alpha(s,p) = -\frac{\partial\mathcal{L}}{\partial p}\,, \qquad \beta(s,p) = \frac{\partial\mathcal{L}}{\partial s}$$

# Map between different formulations

## The relation between single and double potential formulations

The relation between derivatives of Lagrangians in both formulations:

$$g_1 = \frac{2\,\alpha}{\alpha^2 + (\beta+1)^2}\,, \qquad g_2 = \frac{\alpha^2 + \beta^2 - 1}{\alpha^2 + (\beta+1)^2}\,,$$

where $g$ is a function of $\lambda_1, \lambda_2$, which can also be expressed in terms of $\alpha, \beta, s, p$:

$$\lambda_1 = 2\,\alpha\,(1+\beta)\,s - [\alpha^2 - (1+\beta)^2]\,p\,,$$
$$\lambda_2 = [\alpha^2 - (1+\beta)^2]\,s + 2\,\alpha\,(1+\beta)\,p\,,$$

while $w$ is given as:

$$w \equiv \sqrt{\lambda_1^2 + \lambda_2^2} = (\alpha^2 + (\beta+1)^2)\,\sqrt{s^2 + p^2}$$

# Map for duality-symmetric theories

## The $SO(2)$ invariant case

The relation between the two formulations is given in this case by:

$$\frac{\lambda_1}{w}h' = \frac{2\,\alpha}{\alpha^2+(\beta+1)^2}\,, \quad \frac{\lambda_2}{w}h' = \frac{\alpha^2+\beta^2-1}{\alpha^2+(\beta+1)^2}$$

which implies the duality-symmetry condition for the single-potential formulation

$$\beta^2 + \frac{2s}{p}\alpha\beta - \alpha^2 = 1\,,$$

and:

$$(\alpha s + (\beta+1)p)\; h'\Big|_{w=\sqrt{s^2+p^2}(\alpha^2+(\beta+1)^2)} = \alpha\sqrt{s^2+p^2}$$

# Examples: ModMax and BB

## The conformal duality-symmetric electrodynamics

The conformal and duality-symmetric Electrodynamics:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\, \epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

can be translated to single-potential formulation

$$L(s,p) = -\cosh\gamma\, s + \sinh\gamma \sqrt{s^2 + p^2}$$

using a parametrization: $\delta = \coth\frac{\gamma}{2}$. This is so-called ModMax theory. In the special case of $\delta = 1$, the map breaks down. There, the single-field formulation does not exist. This corresponds to Bialynicki-Birula Electrodynamics.

# Example: Generalized Born-Infeld theory

## Generalized Born-Infeld theory

The conventional Lagrangian ($T, \gamma$ are constants):

$$L_{GBI} = \sqrt{UV} - T, \quad U \equiv 2u + e^{\gamma}\,T, \quad V \equiv -2v + e^{-\gamma}\,T\,,$$

where $u \equiv (s + \sqrt{p^2 + s^2})/2$, $v \equiv (-s + \sqrt{p^2 + s^2})/2$.

## Democratic formulation

The duality-symmetric Lagrangian is $\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)$, where in this case $h(w)$ is implicitly given by:

$$h(\lambda) = 4\,T\,\sinh^2\frac{\lambda}{2}\,\cosh(\lambda + \gamma)\,,$$

$$w(\lambda) = -4\,T\,\cosh^2\frac{\lambda}{2}\,\sinh(\lambda + \gamma)\,.$$

# Summary of results

## Main results covered in this talk

- Lorentz covariant formulation for (chiral) p-forms, treating electric and magnetic degrees of freedom on equal footing.
- Lagrangian formulation of arbitrary nonlinear electrodynamics in $3+1$ dimensions treating electric and magnetic degrees of freedom on the same footing.
- Explicit Lagrangian for arbitrary electric-magnetic duality symmetric nonlinear electrodynamics in $3+1$ dimensions.
- Arbitrary non-linear (chiral) p-form (abelian) interactions in the new covariant Lagrangian formulation.
- A simple democratic formulation for type II supergravities.

# Dual theories (free p-forms)

**A $p-$form and its dual**

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F\,, \quad F = dA\,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly particles of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, dual to each other.

**Attention!**

Dual formulations do not admit the same interacting deformations!

# Summary

**Main statements**

- The choice of the free Lagrangian and fundamental variables is important for interacting deformations.
- The Lagrangian can manifest duality symmetry, and more generally, democracy between electric and magnetic degrees of freedom without compromising manifest Poincaré symmetry.
- Abelian self-interactions are simple and tractable in democratic formulation.