Spinning particle dynamics in the worldsheet formalism

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Background

• Spinning particle concept introduced in the context of quasi-classical dynamics of spin [Frenkel, 1926]. Spinning particles are considered as point particles with internal degrees of freedom [Corben,1986; Frydryszak, 1996].

• If a quantum mechanical system realizes an irreducible representation, the classical limit is the dynamical system corresponding to the co-adjoint orbit of the Poincare group (Kirillov, Kostant & Sourieau);

• Models of irreducible spinning particles are known for massless and massive particles in various dimensions. Kuzenko, Lyakhovich, Segal, 1995; Nersessian, Ramos, 1998; Nersessian, Manvelyan, Muller-Kirsten, 2000; Nersessian, 2001; Staruszkiewicz, 2008; Bratek, 2010; Duval, Horvathy, 2015.

• Spinning particle admit consistent couplings with general electromagnetic and gravitational field [Lyakhovich, Sharapov, Segal, 1996]. Equations of motion are studied in uniform magnetic field, and the Coulomb field [Deriglazov, Pupasov-Maksimov, 2014; K., Sinelnikov, 2023].

Particles and surfaces

Example 1. Massive anyon [Gorbunov, Kuzenko, Lyakhovich, 1996]

$$S[x,\varphi] = \int \left(- m\sqrt{-\dot{x}^2\left(1 - \frac{2s}{m}\frac{\dot{\varphi}}{(n,\dot{x})} - \frac{\varrho^2}{m^2}\frac{\dot{\varphi}^2}{(n,\dot{x})^2}\right.} - \varrho\frac{(\partial_\varphi n,\dot{x})}{(n,\dot{x})}\right) d\tau\,. \tag{1}$$

Here, $n = (1, -\cos\varphi, \sin\varphi)$. Solution: general lines on a circular cylinder of radius $r^2 = m^-2(2s^2 + \varrho^2)$ with the tangent vector of symmetry axis p.

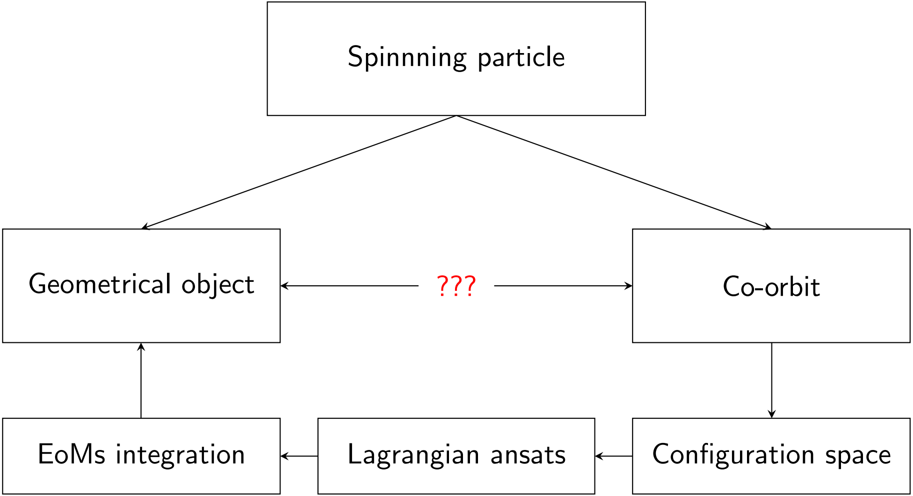
Example 2. Anyon with light-like paths [Nersessian, Ramos, 1998]

$$S[x, e_+] = \int d\sigma(2\alpha - m|dx/d\sigma - e_+|)\,, \qquad \dot{\sigma} = \sqrt{\dot{e}_+}\,, \qquad e_+^2 \approx 0\,. \tag{2}$$

Solution: light-like lines with constant curvature. Such lines lie on a circular cylinder of radius $r = s/m$ with the tangent vector of axis p; $ms = \alpha^2$.

* We use a mostly positive signature of the Minkowski metric.

Problem



Q.: What geometrical objects in Minkowski space correspond to irreducible spinning particles?

Dynamics of irreducible spinning particle

The state of spinning particle is determined by the values of (linear) momentum p and total angular momentum J, being subjected to the mass-shell and spin-shell conditions.

• Both the momentum p and total angular momentum J are integrals of motion for free particle,

$$\dot{p} = 0\,, \qquad \dot{J} = 0\,. \tag{3}$$

• The total angular momentum J is the sum of orbital angular momentum and spin angular momentum,

$$J = x \wedge p + S\,. \tag{4}$$

• The spin angular momentum S is subjected to the set of "Lorentz-spin-shell" conditions [Kaparulin, Lyakhovich, 2017].

Irreducibly of representation and spin

Consider the action of operators of momentum and total angular momentum $\hat{p}$, $\hat{J}$ on the particle positions,

$$\hat{p}_a x^b = i\delta_a{}^b \,, \qquad \hat{J}_{ab} x^c \equiv (x_a \hat{p}_b - x_b \hat{p}_a + \hat{S}_{ab}) x^c = i(x_a \delta_b{}^c - x_b \delta_a{}^c) \,. \tag{5}$$

Once orbital part of angular momentum already generates space-time rotations, the spin angular momentum acts on positions in a trivially,

$$\hat{S}_{ab} f(x^c) = 0 \,, \qquad \forall f(x). \tag{6}$$

Once the representation is irreducible of the Poincare group, any linear operator acting on the space of states, and commuting with $\hat{p}$ and $\hat{J}$, should be a multiple of unit. This applies not only to the elements of the universal enveloping algebra of the Poincare group, but also to any polynomial in the Casimir operators constructed from $\hat{S}$. This conclusion does not depend on the particular choice of internal variables parameterising the configuration space of spin.

Lorents-spin-shell conditions

In d-dimensional space-time, the Lorentz group has $[d/2]$ Casimir operators, being the polynomials in $\hat{S}$.

- In $d = 3$, we have one Casimir operator,

$$C_0(\hat{S}) = \frac{1}{4}\hat{S}_{ab}\hat{S}^{ab}. \tag{7}$$

- In $d = 4$, we have two Casimir operators,

$$C_0(\hat{S}) = \frac{1}{4}\hat{S}_{ab}\hat{S}^{ab}\,, \qquad C_1(\hat{S}) = \frac{1}{4}\varepsilon_{abcd}\hat{S}^{ab}\hat{S}^{cd}\,. \tag{8}$$

- In $d = 5$, we have two Casimir operators,

$$C_0(\hat{S}) = \frac{1}{4}\hat{S}_{ab}\hat{S}^{ab}\,, \qquad C_1(\hat{S}) = \mathrm{tr}(S^4)\,. \tag{9}$$

- In $d = 6$, we have three Casimir operators,

$$C_0(\hat{S}) = \frac{1}{4}\hat{S}_{ab}\hat{S}^{ab}\,, \;\; C_1(\hat{S}) = \frac{1}{4}\varepsilon_{abcdef}\hat{S}^{ab}\hat{S}^{cd}\hat{S}^{ef}\,, \;\; C_2(\hat{S}) = \mathrm{tr}(S^4)\,. \tag{10}$$

Irreducibility of representation and world sheet

In the classical limit, the Casimir operators become Casimir functions, which has to be constant observable.

- In so doing, we get the set of constraints involving spin,

$$C_a(\hat{S}) = \varrho_a \mathrm{id} \qquad \Rightarrow \qquad C_a(S) = \varrho_a\,, \qquad a = 1, \ldots, [d/2]\,. \tag{11}$$

- Since $S = J - x \wedge p$, we get restrictions on the classical particle position,

$$C_a(J - x \wedge p) = \varrho_a\,, \qquad a = 1, \ldots, [d/2]\,. \tag{12}$$

Conclusion

The irreducible spinning particles can be considered as surfaces in space-time. If the world sheet position determines the particle state in a unique way, the set of world sheets is isomorphic to the co-orbit.*

*The opposite is not necessarily true.

Irreducible spinning particle in $3d$ space-time

Consider $3d$ Minkowski space with the coordinates $x^a, a = 0, 1, 2$. Denote by p_a the vector of momentum, and by $J = [x, p] + S$ the vector of particle total angular momentum, being dual of J_{ab}.

Mass-shell and spin-shell conditions:

$$(p, p) + m^2 = 0\,, \qquad (p, J) = ms. \tag{13}$$

Depending on the values of the parameters m, s, three options are possible:

- Massive particle with nonzero (zero) spin: $m \neq 0$, $s \neq 0$ ($s = 0$).
- Spinning particle of continuous helicity: $m = 0$, $ms \neq 0$.
- Massless particle $m = 0$, $s = 0$.

Single Lorentz-spin-shell condition read

$$(S, S) = \varrho^2. \tag{14}$$

Here, we assume that $\varrho \geq 0$.

World sheet classification in $3d$ space-time

The world sheet condition has the form

$$(S, S) \equiv (J - [x, p], J - [x, p]) = \varrho^2. \tag{15}$$

• If m and s are nonzero (massive particle), the world sheet equation determines a circular cylinder of radius $m^2 r^2 = \varrho^2 + s^2$ with a time-like axis,

$$(x, n)^2 + (x - y)^2 = r^2\,, \qquad n = \frac{p}{m}\,, \qquad y = \frac{1}{m^2}[p, J]\,. \tag{16}$$

• If $m = 0$ and $ms = \sigma$ (continuous helicity particle), world sheet equation determines a parabolic cylinder with a light-like axis and focal distance σ,

$$(x, p)^2 + 2(x, v) + a = 0\,, \qquad v = [J, p]\,, \qquad a = J^2 - \varrho^2\,. \tag{17}$$

• If $m = 0$ and $s = 0$ (massless particle), world sheet equation determines a pair of planes with one and the same normal,

$$(x, p)^2 + a = 0\,, \qquad a = J^2 - \varrho^2\,. \tag{18}$$

Note: The space of bi-planes is not isomorphic to the co-orbit.

World lines on world sheets

The spinning particle classical trajectories are general lines $x(\tau)$ lying on a unique representative in the set of world sheets. Using this fact, we need:

- to find a system of ODEs describing general lines that lie on a unique representative in the set of cylindrical surfaces;
- to express the momentum and total angular momentum in terms derivatives of trajectory;
- to describe the gauge symmetries of the model.

Solution: to study differential consequences of world sheet equation.

$$\frac{d^k}{d\tau^k}\Big[(J-[x,p],J-[x,p])-\varrho^2\Big]=0\,,\qquad k=\overline{0,4}. \tag{19}$$

- solving this system with respect to p, J, we express them in terms of derivatives of path;
- the consistency condition gives the equations of motion;
- the gauge symmetries are shifts along the world sheet.

Geometry of light-like and time-like curves

For a time-like path $x(\tau)$ parameterised by the natural parameter such that $(\dot{x}, \dot{x}) = 1$ the Frenet frame reads

$$T = \dot{x}\,, \qquad N = \frac{\ddot{x}}{\sqrt{(\ddot{x}, \ddot{x})}}\,, \qquad B = \frac{[\dot{x}, \ddot{x}]}{\sqrt{(\ddot{x}, \ddot{x})}}. \tag{20}$$

The Frenet formulas have the form:

$$\dot{T} = \varkappa B\,, \qquad \dot{N} = -\varkappa T + \omega B\,, \qquad \dot{B} = \omega N. \tag{21}$$

For a light-like path $x(\tau)$ parameterised by the natural parameter such that $(\ddot{x}, \ddot{x}) = 1$ the Frenet frame reads,

$$T = \dot{x}\,, \qquad N = \ddot{x}\,, \qquad B = \dddot{x} + \frac{1}{2}(\dddot{x}, \dddot{x})\dot{x}. \tag{22}$$

The Frenet formulas have the form:

$$\dot{T} = B\,, \qquad \dot{N} = -T + \varkappa B\,, \qquad \dot{B} = \varkappa N. \tag{23}$$

$\varkappa$ is the curvature, ω is the torsion.

Time-like paths of massive particles

Expressions for p, J

$$p = \frac{m}{\sqrt{z^2 - 1}}\left(zT + \frac{\dot{\varkappa}}{\varkappa^2}\frac{1}{3z + r\omega(z^2-1)}N + \frac{1}{r\varkappa}\frac{1}{z^2-1}B\right); \tag{24}$$

$$J = [x, p] + \frac{sr}{m}\left(\frac{\dot{\varkappa}}{\varkappa^2}\frac{1}{3z + r\omega(z^2-1)}B - \frac{1}{r\varkappa}\frac{1}{z^2-1}N\right). \tag{25}$$

The solution uses z, being the common root of polynomials $P(z)$, $Q(z)$.

Dynamical equation

Auxiliary polynomials:

$$P(z) = r^4\varkappa^4\omega^2 z^8 + \dots, \qquad Q(z) = 3r^3\varkappa^4\omega z^8 + \dots; \tag{26}$$

EoM in implicit form:

$$\mathrm{Res}_z(P(z), Q(z)) = 0\,. \tag{27}$$

Special case $r\varkappa, r\omega \ll 1$

Expressions for p, J

$$p = m\left(T + r\varkappa\frac{\dddot{\varkappa}\varkappa\omega - 3\ddot{\varkappa}^2 - \dot{\varkappa}(2\dot{\varkappa}\omega + \varkappa\dot{\omega})}{4\dot{\varkappa}^2 - 3\ddot{\varkappa}\varkappa}N\right); \tag{28}$$

$$J = [x, p] + \frac{s}{m}\left(\frac{2r\dot{\varkappa}\varkappa\omega - 3r\varkappa^2\dot{\omega} + 9\varkappa^3}{4\dot{\varkappa}^2 - 3\ddot{\varkappa}\varkappa}N + r\varkappa B\right). \tag{29}$$

The solution is well-defined if $4\dot{\varkappa}^2 - 3\ddot{\varkappa}\varkappa \neq 0$.

Dynamical equations

Auxiliary polynomials:

$$P(z) = 2\varkappa^2\omega z^3 + \dots, \qquad Q(z) = 4\varkappa^2\omega z^4 + \dots. \tag{30}$$

EoM in implicit form:

$$\mathrm{Res}_z(P(z), Q(z)) = 0. \tag{31}$$

Light-like paths of massive particles

Expressions for p, J

$$p = m(r^{\frac{1}{2}}\dddot{x} + r^{-\frac{1}{2}}\dot{x}); \tag{32}$$

$$J = [x + r\ddot{x}, p] + s(r^{\frac{1}{2}}\dddot{x} + r^{-\frac{1}{2}}\dot{x})\,. \tag{33}$$

Here, r is the cylinder radius.

Dynamical equation

$$\varkappa = \frac{1}{r} \tag{34}$$

The light-like cylindrical lines are helices with light-like tangent vector.

The model has been known previously Nersessian, Ramos, 1999; For $d = 4$ spacetime, see Nersessian, Ramos, 1998. The generalisation of the model for higher dimensions is not known.

Time-like paths of continuous helicity particles

Expressions for p, J

$$p=\sqrt{-\mathrm{sgn}(\sigma)\sigma\varkappa z}\left(T-\frac{\dot{\varkappa}\alpha}{3\varkappa^2 z-\varkappa\omega}N+zB\right); \tag{35}$$

$$J=\left[\frac{(x,p)^2\varkappa z+\sigma}{2\sqrt{-\mathrm{sgn}(\sigma)\sigma\varkappa z}}+\frac{a+\varrho}{2}\sqrt{\frac{-\varkappa z}{\mathrm{sgn}(\sigma)\sigma}}\right]T+\ldots\,, \qquad a=\ldots \tag{36}$$

The solution uses z, being the common root of polynomials $P(z)$, $Q(z)$.

Dynamical equations

$$P(z)=9z^3+9\varkappa^{-2}\dot{\varkappa}z^2+\ldots\,, \qquad Q(z)=9z^4+6\varkappa^{-2}\dot{\varkappa}z^3+\ldots\,. \tag{37}$$

$$\mathrm{Res}_z(P(z),Q(z))=0\,. \tag{38}$$

Kaparulin, Lyakhovich, Retuntsev, 2022.

Light-like paths of continuous helicity particles

Expressions for p, J

$$p = \sqrt{\sigma}\,\dddot{x}\,; \tag{39}$$

$$J = -\sqrt{\sigma}\Big[\dot{x} + (x,\,\dddot{x})\ddot{x} - \Big((x,\ddot{x}) + \frac{\varrho}{2\sigma}\Big)\,\dddot{x}\Big]\,. \tag{40}$$

Dynamical equations

$$\varkappa = 0\,. \tag{41}$$

The light-like paths on parabolic cylinder have zero curvature.

The construction does not work because the space of light-like paths has higher dimension than the co-orbit.

Kaparulin, Lyakhovich, Retuntsev, 2022.

What about massless particles?

The world sheets of massless particles are bi-planes:

$$(x,p)^2 - a = 0\,, \qquad a > 0. \tag{42}$$

The set of bi-planes is parameterised by three parameters (2 for p and one for a. So, the world sheet position keeps particular information about the particle state.

• We have no isomorphism

$$\text{World sheets} \qquad \Leftrightarrow \qquad \text{Co-orbit points}\,. \tag{43}$$

• Nevertheless, we can say that the spinning particle path are planar curves. This fact has been observed in previous works, eg. Ramos, Roca, 1995; Duval, Horvathy, 2015; Duval, Elbistan, Horvathy, Zhang, 2015.

Conclusion and outlook

Results:

• It has been shown that the irreducible spinning particles can be considered as geometrical objects (surfaces, lines). The topology of the surface is determined by the representation.

• Considering the spinning particle trajectories as general lines on the world sheet, we derived the geometrical equations of motion, and expressed momentum and total angular momentum in terms of derivatives of path.

• It has been shown that the trajectories of massless particles are planar curves.

Further studies:

• Particles on gravitational or EM background.

• Massless particle trajectories.