Higher Schwarzians

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Main result

We revisit the construction of supersymmetric Schwarzians and their higher order variants using the nonlinear realizations. We show that the Schwarzians can be systematically obtained as certain projections of Maurer-Cartan forms of (super)conformal groups after imposing simple conditions on them. Likewise, we also present the supersymmetric Schwarzian actions, defined as the integrals of products of Cartan forms. In contrast with the previous attempts to obtain the super-Schwarzians within nonlinear realizations technique, our set of constraints do not impose any restriction on the super-Schwarzians.

Plan

- Introduction
- Three steps towards Schwarzian. $\mathcal{N}=0$ case
- New version of the bosonic Schwarzians
- $\mathcal{N}=2$ supersymmetric Schwarzian
- Schwarzians of the higher order
- Conclusion

The Schwarzian derivative $\{t,\tau\}$ defined as

$$\{t,\tau\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2, \quad \dot{t} = \partial_\tau t, \tag{1}$$

itself appears in seemingly unrelated fields of physics and mathematics. The action of the bosonic Schwarzian mechanics reads

$$S_{schw}[t] = -\frac{1}{2}\int d\tau \left(\{t,\tau\} + 2m^2\dot{t}^2\right).$$

Remarkably, the equation of motion of this higher-derivative action is equivalent just to

$$\frac{d}{d\tau}\left[\{t,\tau\} + 2m^2\dot{t}^2\right] = 0.$$

The characteristic feature of the Shwarzian derivative (1) is its invariance under $Sl(2,\mathbb{R})$ Möbius transformations acting on $t[\tau]$ via

$$t \to \frac{at+b}{ct+d}.$$

The presence of $m^2\dot{t}^2$ term in the action $S_{schw}[t]$ modify the realization of $Sl(2,\mathbb{R})$ symmetry. The simplest way to understand the modification is to notice that the action $S_{schw}[t]$ can be represented as

$$S_{schw}[t] = -\frac{1}{2}\int d\tau\,\{F,\tau\}\,, \qquad F[\tau] = \tan\left(mt[\tau]\right), \tag{2}$$

and, therefore, the action (2) possesses the $Sl(2,\mathbb{R})$ invariance via

$$F \to \frac{aF+b}{cF+d}$$

with $F[\tau]$ defined in (2).
Being invariant under $d=1$ conformal transformation, the Schwarzian derivative naturally appears in the transformations of the conformal stress tensor $T(z)$ (A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, 1984)

$$T(z) = \left(\frac{d\tilde{z}}{dz}\right)^2 \tilde{T}(\tilde{z}) + \{\tilde{z}, z\}\,. \tag{3}$$

The $\mathcal{N}{=}1,2,3,4$ supersymmetric generalization of the Schwarzian derivative are present in the transformation properties of the current superfield $J^{(\mathcal{N})}(Z)$ generating $\mathcal{N}$ - extended superconformal transformations (K. Schoutens,1988). Thus, we have complete zoo of the supersymmetric Schwarzians.

Although the Schwarzian derivative is a third order nonlinear differential operator, the general solution to the equations

$$\{t,\tau\} = v(\tau) \qquad (*)$$

has a very nice description. Let $g_1(\tau), g_2(\tau)$ be two linearly independent solutions to the linear equation

$$2g'' + v(\tau)g = 0$$

then the solution to the equation (*) is given by

$$t(\tau) = g_1(\tau)/g_2(\tau)$$

and any solution is of this form.

The treatment of the supersymmetric Schwarzians as the anomalous terms in the transformations of the currents superfield $J^{(\mathcal{N})}(Z)$ leads to the conclusion that the structure of the (super)Schwarzians is completely defined by the conformal symmetry and, therefore, it should exist a different, probably purely algebraic, way to define the (super)Schwarzians. The main property of the (super)Schwarzians which define their structure, is their invariance with respect to (super)conformal transformations. The suitable way to construct (super)conformal invariants is the method of nonlinear realizations equipped by the inverse Higgs phenomenon. Such approach, demonstrated how the Schwarzians can be obtained via the non-linear realizations approach, was initiated in Anton Galajinsky paper (A. Galajinsky, 2019) and then it was applied to different super-conformal algebra in the series of his papers.Later on, this approach has been extended to the cases of non-relativistic Schwarzians and Carroll algebra(J. Gomis, D. Hidalgo, P. Salgado-Rebolledo, 2021).
The preference of the non-linear realizations approach for construction of the Schwarzians with respect to approach related with superconformal transformations, is much more wide area of its applications. Indeed, the non-linear realization method works perfectly for any (super)algebra and the set of invariant Cartan forms can be easily obtained.

Thus, the main questions in such approach are

- What is the role and source of the "boundary" time τ and its supersymmetric partners?
- Which constraints have to be imposed on the Cartan forms? Which forms nullified and how to construct the action from the surviving forms?
- Which additional technique can be used to simplify the calculations?

Of course, these questions was already partially analyzed and answered. However, some important properties and statements were missing. Moreover, the constraints proposed in the previous papers looks like the results of illuminating guess. The main puzzle is the fact that the constraints were imposed on the fermionic projections of the forms, but not on the forms themselves. Thus, the questions why it is so and what happens with the full Cartan forms after imposing of such constraints have been not fully analyzed. Finally, in the cases of more complicate superconformal group the calculations quickly become a rather cumbersome and the standard technique does not help.

Step one

The bosonic conformal group in $d=1$ is infinite-dimensional. Its finite dimensional $sl(2,\mathbb{R})$ subalgebra spanned by the Hermitian generators of translation P, dilatation D and conformal boost K, can be fixed by the following relations

$$\mathrm{i}\,[D,P]=P,\quad \mathrm{i}\,[D,K]=-K,\quad \mathrm{i}\,[K,P]=2D.$$

If we parameterized the $SL(2,\mathbb{R})$ - group element g as

$$g=e^{\mathrm{i}t\left(P+m^2K\right)}e^{\mathrm{i}zK}e^{\mathrm{i}uD},$$

then the Cartan forms

$$g^{-1}dg=\mathrm{i}\omega_P P+\mathrm{i}\omega_D D+\mathrm{i}\omega_K K$$

read

$$\omega_P=e^{-u}dt,\quad \omega_D=du-2zdt,\quad \omega_K=e^{u}\left(dz+z^2dt+m^2dt\right).$$

The infinitesimal $sl(2,\mathbb{R})$ transformations

$$g\ \rightarrow\ g'=e^{\mathrm{i}aP}\ e^{\mathrm{i}bD}\ e^{\mathrm{i}cK}\ g$$

leaving the Cartan forms invariant read

$$\delta t=a\,\frac{1+\cos(2mt)}{2}+b\,\frac{\sin(2mt)}{2m}+c\,\frac{1-\cos(2mt)}{2m^2},\quad \delta u=\frac{d}{dt}\delta t,\quad \delta z=\frac{1}{2}\frac{d}{dt}\delta u-\frac{d}{dt}\delta t\,z.$$

Step two

All Cartan forms are invariant with respect to $sl(2,\mathbb{R})$ transformations. Notice, within the nonlinear realization approach we implicitly mean that the "coordinates" u and z are functions depending on time t. However, neither "time" t, nether its differentials dt are invariant under $sl(2,\mathbb{R})$ transformations. Thus, to get the invariants one has to introduce the "invariant time" τ and parameterize the form ω_P as

$$\omega_P = e^{-u}dt = d\tau \quad \Rightarrow \quad \dot{t} = e^u,\ \dot{u} = \frac{\ddot{t}}{\dot{t}},\ \ddot{u} = \frac{\dddot{t}}{\dot{t}} - \left(\frac{\ddot{t}}{\dot{t}}\right)^2 .$$

Let us stress again that the τ is a new "invariant time" which completely inert under $sl(2,\mathbb{R})$ transformations. Correspondingly, the rest $sl(2,\mathbb{R})$ forms now read

$$\omega_D = \left(\dot{u} - 2e^u z\right) d\tau, \quad \omega_K = e^u \left(\dot{z} + e^u \left(z^2 + m^2\right)\right) d\tau.$$

Now, nullifying the form ω_D we will express the field $z(\tau)$ in terms of dilaton $u(\tau)$ and then in terms of new time τ

$$\omega_D = 0 \quad \Rightarrow \quad z = \frac{1}{2}e^{-u}\dot{u} = \frac{\ddot{t}}{2\dot{t}^2}.$$

This is particular case of the Inverse Higgs phenomenon (E.A. Ivanov, V.I. Ogievetsky, 1975).

Step three

After Second step we are leaving with only one field - "old time" $t(\tau)$ and only one invariant - form ω_K which now reads (The form $\omega_P = d\tau$ is also invariant. However, adding this form to the action evidently does not produce new equations of motion)

$$\omega_K = \frac{1}{2}\left[\ddot{u} - \frac{1}{2}\dot{u}^2 + 2m^2 e^{2u}\right] d\tau = \frac{1}{2}\left[\frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2 + 2m^2 \dot{t}^2\right] d\tau$$

Thus, the Schwarzian action (2) can be re-obtained within our approach as

$$\mathcal{S}[t] = -\int \omega_K .$$

It proves useful to re-write the form ω_K and, therefore, the Schwarzian action in terms of dilaton $u(t)$ and "old time" variable t

$$\mathcal{S}[u] = -\int \omega_K = \int dt \left(\left(\frac{dy}{dt}\right)^2 - m^2 y^2\right), \qquad y(t) = e^{\frac{1}{2}u(t)}.$$

Thus, formally speaking, the action of Schwarzian mechanics is just the action of one dimensional harmonic oscillator rewritten in terms of time variable t depending on new inert time variable τ.

As the first example of the application of the proposed approach, let us consider the nonlinear realization of the Maxwell algebra in $d = 1$.
The Maxwell algebra contains the Hermitian generators of translation P, analogue of the dilatation - central charge generator Z, analogue of the conformal boost K, and the generator of $U(1)$ rotations obeying the following relations

$$\mathrm{i}\,[J,P] = P, \quad \mathrm{i}\,[J,K] = -K, \quad \mathrm{i}\,[K,P] = 2Z.$$

If we parameterized the Maxwell - group element g as

$$g = e^{\mathrm{i}t\left(P+qJ+m^2K\right)}e^{\mathrm{i}zK}e^{\mathrm{i}uZ}e^{\mathrm{i}\phi J},$$

then the Cartan forms read

$$\omega_P = e^{-\phi}dt, \quad \omega_Z = du - 2zdt, \quad \omega_K = e^{\phi}\left(dz - qzdt + m^2dt\right), \quad \omega_J = d\phi.$$

The constraints

$$\omega_P = d\tau, \quad \omega_Z = 0$$

result in the following relations

$$\dot{t} = e^{\phi}, \quad z = \frac{\dot{u}}{2\dot{t}}.$$

Finally,

$$\omega_K = \dot{t}\left[\frac{1}{2}\left(\frac{\ddot{u}}{\dot{t}} - \frac{\dot{u}\ddot{t}}{\dot{t}^2}\right) + m^2\dot{t} - \frac{1}{2}q\dot{u}\right].$$

This is exactly flat space analogue of the Schwarzian constructed in H. Afshar and H.A. Gonzalez, D. Grumiller, D. Vassilevich, 2020.

The $\mathcal{N}$=2 super-Schwarzian has been introduced in J.D. Cohn,*N = 2 super Riemann surfaces*, (1987) and then it was re-obtaines in K. Schoutens, *O(N)-Extended superconformal field theory in superspace*, (1988) . The treatment of the $\mathcal{N}$=2 super-Schwarzian within the nonlinear realization of the $su(1,1|1)$ supergroup was initiated in A. Galajinsky, *Super-Schwarzians via nonlinear realizations*, (2020). The consideration performed in this paper correctly reproduced $\mathcal{N}$=2 super-Schwarzian but unfortunately the constraints used there imposed the further constraint on the super-Schwarzian to be a constant. Now, I will demonstrate that our variant of the constraints correctly reproduce $\mathcal{N}$=2 super-Schwarzian, expressed all $su(1,1|1)$ Cartan forms in terms of this super-Schwarzian and its derivatives. Finally, we will show that imposing the constraints on the full Cartan forms makes possible to utilize the Maurer-Cartan equations which drastically simplify all calculations.

In the case of $\mathcal{N}=2$ supersymmetry we are dealing with the $\mathcal{N}=2$ superconformal algebra $su(1,1|1)$ defined by the following relations

$$\mathrm{i}\,[D,P]=P,\quad \mathrm{i}\,[D,K]=-K,\quad \mathrm{i}\,[K,P]=2D,$$

$$\left\{Q,\overline{Q}\right\}=2P,\quad \left\{S,\overline{S}\right\}=2K,\quad \left\{Q,\overline{S}\right\}=-2D+2J, \left\{\overline{Q},S\right\}=-2D-2J,$$

$$\mathrm{i}\,[J,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[J,\overline{Q}\right]=-\frac{1}{2}\overline{Q},\quad \mathrm{i}\,[J,S]=\frac{1}{2}S,\ \mathrm{i}\left[J,\overline{S}\right]=-\frac{1}{2}\overline{S},$$

$$\mathrm{i}\,[D,Q]=\frac{1}{2}Q,\ \mathrm{i}\left[D,\overline{Q}\right]=\frac{1}{2}\overline{Q},\quad \mathrm{i}\,[D,S]=-\frac{1}{2}S,\ \mathrm{i}\left[D,\overline{S}\right]=-\frac{1}{2}\overline{S},$$

$$\mathrm{i}\,[K,Q]=-S,\ \mathrm{i}\left[K,\overline{Q}\right]=-\overline{S},\quad \mathrm{i}\,[P,S]=Q,\ \mathrm{i}\left[P,\overline{S}\right]=\overline{Q}.$$

Defining the "inert" element $g_0 = e^{\mathrm{i}\tau P}\, e^{\theta Q + \bar\theta \overline{Q}}$ and calculating the "inert" Cartan forms

$$\Omega_0 = g_0^{-1} dg_0 = \mathrm{i}\left(d\tau - \mathrm{i}(\theta d\bar\theta + \bar\theta d\theta)\right) P + d\theta Q + d\bar\theta \overline{Q} \equiv \mathrm{i}\triangle\tau P + d\theta Q + d\bar\theta\overline{Q},$$

one may easily construct the covariant derivatives

$$\mathcal{D}_\tau = \partial_\tau,\ \mathcal{D} = \frac{\partial}{\partial\theta} - i\bar\theta\frac{\partial}{\partial\tau},\ \overline{\mathcal{D}} = \frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial\tau}, \qquad \{\mathcal{D}, \overline{\mathcal{D}}\} = -2i\partial_\tau$$

Thus, from now we will treated all fields as the superfields depending on the coordinates of "inert" superspace $\{\tau, \theta, \bar\theta\}$.
Similarly to the previously considered cases, we choose the following parametrization of the general element of the $\mathcal{N}$=2 superconformal group $SU(1,1|1)$

$$g = e^{it(P+m^2K)}\, e^{\xi Q + \bar\xi\overline{Q}}\, e^{\psi S + \bar\psi \overline{S}} e^{izK} e^{iuD} e^{\phi J}$$

where the parameters $t, \xi, \bar\xi, \psi, \bar\psi, z, u$ and ϕ are, as we stated above, the superfunctions depending on $\{\tau, \theta, \bar\theta\}$.
The Cartan forms

$$g^{-1}dg = i\omega_P P + \omega_Q Q + \bar\omega_Q \overline{Q} + i\omega_D D + \omega_J J + \omega_S S + \bar\omega_S \overline{S} + i\omega_K K$$

explicitly read

$$\omega_P \equiv e^{-u}\triangle t = e^{-u}\left(dt - i(\xi d\bar\xi + \bar\xi d\xi)\right),$$
$$\omega_Q = e^{-\frac{u}{2}+i\frac{\phi}{2}}\left(d\xi + \psi\,\triangle t\right),\; \bar\omega_Q = e^{-\frac{u}{2}-i\frac{\phi}{2}}\left(d\bar\xi + \bar\psi\,\triangle t\right),$$
$$\omega_D = du - 2z\,\triangle t - 2i(d\xi\bar\psi + d\bar\xi\psi),\; \omega_J = d\phi - 2\psi\bar\psi\,\triangle t + 2(d\bar\xi\psi - d\xi\bar\psi) - 2m^2\xi\bar\xi dt$$
$$\omega_S = e^{\frac{u}{2}+i\frac{\phi}{2}}\left(d\psi - i\psi\bar\psi d\xi + z\,(d\xi + \psi\,\triangle t)\; - m^2\,(1 - \mathrm{i}\bar\xi\psi)\,\xi dt\right),$$
$$\bar\omega_S = e^{\frac{u}{2}-i\frac{\phi}{2}}\left(d\bar\psi + i\psi\bar\psi d\bar\xi + z\,(d\bar\xi + \bar\psi\,\triangle t)\; - m^2\,(1 - \mathrm{i}\xi\bar\psi)\,\bar\xi dt\right),$$
$$\omega_K = e^{u}\left(dz + z^2\triangle t - i(\psi\,d\bar\psi + \bar\psi\,d\psi) + 2iz\,(d\xi\,\bar\psi + d\bar\xi\psi) + m^2\,\left(1 + \mathrm{i}\,(\psi\bar\xi + \bar\psi\xi)\right)^2\,dt\right).$$

Now,identifying the forms $\omega_P, \omega_Q, \bar\omega_Q$ with $\triangle\tau$, $d\theta$ and $d\bar\theta$ we will get the following equations

$$e^{-u}\triangle t = e^{-u}\left(dt + \mathrm{i}\left(d\bar\xi\xi + d\xi\bar\xi\right)\right) = \triangle\tau \quad\Rightarrow\quad \begin{cases} \dot t + \mathrm{i}\left(\dot{\bar\xi}\xi + \dot\xi\bar\xi\right) = e^{u}, \\ Dt + \mathrm{i}D\xi\,\bar\xi = 0, \\ \overline{D}t + \mathrm{i}\overline{D}\bar\xi\,\xi = 0, \end{cases}$$

$$e^{-\frac12(u-\mathrm{i}\,\phi)}\left(d\xi + \psi\triangle t\right) = d\theta \quad\Rightarrow\quad \begin{cases} \dot\xi + e^{u}\psi = 0, \\ D\xi = e^{\frac12(u-\mathrm{i}\,\phi)}, \\ \overline{D}\xi = 0, \end{cases}$$

$$e^{-\frac12(u+\mathrm{i}\,\phi)}\left(d\bar\xi + \bar\psi\triangle t\right) = d\bar\theta \quad\Rightarrow\quad \begin{cases} \dot{\bar\xi} + e^{u}\bar\psi = 0, \\ \overline{D}\bar\xi = e^{\frac12(u+\mathrm{i}\,\phi)}, \\ D\bar\xi = 0. \end{cases}$$

Finally, one has to nullify the form ω_D :

$$\omega_D = du - 2e^u z \triangle\tau - 2i(e^{\frac{1}{2}(u-\mathrm{i}\,\phi)} d\theta\bar\psi + e^{\frac{1}{2}(u+\mathrm{i}\,\phi)} d\bar\theta\psi) = 0 \quad \Rightarrow \quad \begin{cases} \dot u - 2e^u z = 0, \\ Du = 2\mathrm{i}\, e^{\frac{1}{2}(u-\mathrm{i}\,\phi)}\bar\psi, \\ \overline{D}u = 2\mathrm{i}\, e^{\frac{1}{2}(u+\mathrm{i}\,\phi)}\psi. \end{cases}$$

From these relations one may obtained several important consequences. In particular, we have

$$Du = \mathrm{i}D\phi, \overline{D}u = -\mathrm{i}\overline{D}\phi, \quad \Rightarrow \quad \left[D, \overline{D}\right] u = -2\dot\phi, \; \left[D, \overline{D}\right]\phi = 2\dot u,$$

$$D\bar\psi = 0, \quad \overline{D}\psi = 0, \quad \psi = -\frac{\dot\xi}{D\xi\overline{D}\bar\xi}, \; \bar\psi = -\frac{\dot{\bar\xi}}{D\xi\overline{D}\bar\xi},$$

$$D\xi\,\overline{D}\bar\xi = e^u, \; \dot u = \frac{D\dot\xi}{D\xi} + \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi}, \quad \frac{\overline{D}\bar\xi}{D\xi} = e^{\mathrm{i}\phi}, \quad \dot\phi = \mathrm{i}\left(\frac{D\dot\xi}{D\xi} - \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi}\right).$$

Now, one may check that the form ω_J reads

$$\omega_J = \mathrm{i}\left[\frac{D\dot\xi}{D\xi} - \frac{\overline{D}\dot{\bar\xi}}{\overline{D}\bar\xi} - 2\mathrm{i}\frac{\dot\xi\dot{\bar\xi}}{D\xi\overline{D}\bar\xi} + 2\mathrm{i}m^2\xi\bar\xi D\xi\overline{D}\bar\xi\right]\triangle\tau \equiv \mathrm{i}\triangle\tau\,\mathcal{S}_{\mathcal{N}=2}.$$

Thus we see, that $\mathcal{N}=2$ Schwarzian $\mathcal{S}_{\mathcal{N}=2}$ appears automatically.

One may check that the other Cartan forms, $\omega_S, \bar{\omega}_S$ and ω_K can be also expressed in terms of the $\mathcal{N}$=2 Schwarzian only

$$\begin{aligned}
\omega_P &= \triangle\tau, \omega_Q = d\theta,\ \bar{\omega}_Q = d\bar{\theta}, \quad \omega_J = \mathrm{i}\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \\
\omega_S &= -\frac{1}{2}\mathcal{S}_{\mathcal{N}=2}\, d\theta - \frac{\mathrm{i}}{2}\overline{D}\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \quad \bar{\omega}_S = \frac{1}{2}\mathcal{S}_{\mathcal{N}=2}\, d\bar{\theta} + \frac{\mathrm{i}}{2}D\mathcal{S}_{\mathcal{N}=2}\triangle\tau, \\
\omega_K &= \frac{1}{2}D\mathcal{S}_{\mathcal{N}=2}d\theta - \frac{1}{2}\overline{D}\mathcal{S}_{\mathcal{N}=2}d\bar{\theta} + \frac{1}{4}\left(\mathrm{i}\left[D, \overline{D}\right]\mathcal{S}_{\mathcal{N}=2} - \mathcal{S}^2_{\mathcal{N}=2}\right)\triangle\tau.
\end{aligned}$$

Thus one can expect that the proper Schwarzian action reads

$$S_{N2schw} = -\frac{\mathrm{i}}{2}\int d\tau\, d\theta\, d\bar{\theta}\mathcal{S} = -\frac{1}{2}\int \omega_J \wedge \omega_Q \wedge \bar{\omega}_Q = \mathrm{i}\int \omega_P \wedge \omega_S \wedge \bar{\omega}_Q.$$

In 1969 D. Aharonov gave definitions of higher-order analogues of the Schwarzian derivative. Later on, Tamanoi introduced another set of higher order Schwarzian derivatives. Finally, Kim and Sugawa derived relations between the Aharonov invariants and Tamanoi's Scwarzian derivatives.
Both these definitions leads to higher order Schwarzian derivatives invariant with respect to $sl(2, R)$ transformations and both definitions are non geometric ones.
Using the results from the previous Section, one may propose purely geometric definition of the higher Schwarzians. Let us start with the semi-positive central-charge less subalgebra $L_n, n \geq -1$ of Virasoro algebra

$$\mathrm{i}\,[L_n, L_m] = (n-m)L_{n+n}, \qquad n, m \geq -1. \tag{4}$$

It is clear, that the $sl(2, R)$ subalgebra can be defined within (4) as

$$P = L_{-1},\; D = L_0,\; K = L_1. \tag{5}$$

Let us now parametrized the element g of the Virasoro group as

$$g = e^{\mathrm{i}\,tL_{-1}}\, e^{\mathrm{i}\,z_1 L_1}\, e^{\mathrm{i}\,uL_0} \prod_{i=2}^{\infty} e^{\mathrm{i}z_i L_i} \tag{6}$$

It is completely clear that all higher parameters z_i are invariant with respect to $sl(2,R)$ transformations realized by the left multiplication of the element g by the element $g_0 \in sl(2,R)$

$$g_0\, g = g' \;\Rightarrow\; \delta t = \tilde{a} + \tilde{b}t + \tilde{c}t^2, \;\; \delta u = \frac{d}{dt}\delta t, \quad \delta z = \frac{1}{2}\frac{d}{dt}\delta u - \frac{d}{dt}\delta t\, z, \;\; \delta z_i = 0, \;\; i \geq 2. \qquad (7)$$

Thus, all parameters $z_i, i \geq 2$ are good candidates to be the higher Schwarzians.
To provide a proper parametrization of the parameters z_i in terms of parameter $t[\tau]$ depending on the inert time τ, one has , by analogy with the previous cases, impose the constraints on the Cartan forms ω_i

$$\Omega = g^{-1}dg = \mathrm{i} \sum_{n=-1} \omega_n L_n. \qquad (8)$$

Note, that by construction all Cartan forms in (8) are invariant under $sl(2,R)$ transformations (7).
Let us list several first Cartan forms :

$$\begin{aligned}
\omega_{-1} &= e^{-u} dt, \\
\omega_0 &= du - 2z_1\, dt, \\
\omega_1 &= e^{u}\left(dz_1 + z_1^2 dt\right) - 3e^{-u} z_2\, dt, \\
\omega_2 &= dz_2 - 2z_2 du + 4\, dt\, z_1 z_2 - 4\, e^{-u} z_3\, dt, \\
\omega_3 &= dz_3 - e^{u} z_2 dz_1 - e^{u} z_1^2 z_2\, dt + \frac{3}{2} e^{-u} z_2^2\, dt - 3z_3 du + 6z_1 z_3\, dt - 5e^{-u} z_4\, dt, \;\; etc. \qquad (9)
\end{aligned}$$

The final step is to impose the following constraints

$$\omega_{-1} = d\tau, \quad \omega_n = 0, \quad n \geq 0. \tag{10}$$

As the result, we will obtained the following expressions for the parameters z_n

$$\begin{aligned}
z_1 &= \frac{t''}{2(t')^2}, \\
z_2 &= \frac{1}{6}\left[\frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2\right] = \frac{1}{6}\{t,\tau\} = \frac{1}{6}S_t, \\
z_3 &= \frac{1}{24}S_t', \\
z_4 &= \frac{1}{120}\left(S_t'' - S_t^2\right), \\
z_5 &= \frac{1}{720}\left(S_t^{(3)} - 2S_tS_t'\right), \\
z_6 &= \frac{1}{30240}\left(6S_t^{(4)} - 12S_tS_t'' - 27(S_t')^2 - 20S_t^3\right), \ etc
\end{aligned} \tag{11}$$

One may compare our set of higher Schwarzianswith Aharonov's invariants and Tamanoi Schwarzians

Aharonov's invariants ψ_k | Tamanoi's Schwarzians s_k

$$\frac{t'[\tau]}{(t[\tau+w]-t[\tau])}=\frac{1}{w}-\sum_{k=0}\psi_{k+1}w^k \qquad \frac{t'[\tau](t[\tau+w]-t[\tau])}{\frac{1}{2}t''[\tau](t[\tau+w]-t[\tau])+t'[\tau]^2}=\sum_{k=1}w^k s_k$$

$$\psi_1=\frac{t''}{2t'} \qquad s_1=1,\ s_2=0$$

$$\psi_2=\frac{1}{6}S_t \qquad s_3=\frac{1}{6}S_t$$

$$\psi_3=\frac{1}{24}S'_t \qquad s_4=\frac{1}{24}S'_t$$

$$\psi_4=\frac{1}{360}\left(3S''_t+2S_t^2\right) \qquad s_5=\frac{1}{120}\left(S''_t+4S_t^2\right)$$

$$\psi_5=\frac{1}{720}\left(S'''_t+3S_tS'_t\right) \qquad s_6=\frac{1}{720}\left(S'''_t+13S_tS'_t\right)$$

$$\psi_6=\frac{1}{60480}\left(12S_t^{(4)}+60S_tS''_t+51(S'_t)^2+16S_t^3\right) \qquad s_7=\frac{1}{5040}\left(S_t^{(4)}+19S_tS''_t+13(S'_t)^2+34S_t^3\right). \tag{12}$$

Thus we see, that the difference in the Schwarzians appears already at four order.

Comment 1
Note, that our constraints are invariant with respect to whole Virasoro algebra. If we will consider the left multiplication of our group element g (6) by the element $g_n = e^{a_n L_n}$

$$g_n g = g' \tag{13}$$

then we will obtain

$$\delta_n t = a_n t^{n+1}, \quad \delta_n S_t = a_n n(n^2-1) Y_n, \quad \delta Y_k = a_n (k+2n) Y_{k+n}, \tag{14}$$

where

$$Y_k = t^{k-2}(t')^2. \tag{15}$$

It is clear that the variation of the higher Scwarzians will provide us with the new set of the higher Scwarzians which contain both S and Y^n and their derivatives. The usefulness of this full Virasoro symmetry is not clear for us yet.

Comment 2

It should be clear that the parameterization of the Virasoro group element g in (6) is not unique. For example, the next obvious parameterization reads

$$\tilde{g} = e^{\mathrm{i}\, tL_{-1}}\, e^{\mathrm{i}\, \tilde{z}_1 L_1}\, e^{\mathrm{i}\, uL_0}\;\, e^{\sum_{i=2}^{\infty} \mathrm{i}\tilde{z}_i L_i}. \qquad (16)$$

The same constraints (10) on the Cartan forms lead to the following expressions for the several first parameters $\tilde{z}_n$

$$\begin{aligned}
\tilde{z}_1 &= \frac{t''}{2(t')^2}, \\
\tilde{z}_2 &= \frac{1}{6}\left[\frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2\right] = \frac{1}{6} S_t, \\
\tilde{z}_3 &= \frac{1}{24} S_t', \\
\tilde{z}_4 &= \frac{1}{120}\left(S_t'' - S_t^2\right), \\
\tilde{z}_5 &= \frac{1}{1440}\left(2\, S_t^{(3)} - 9 S_t S_t'\right), \\
\tilde{z}_6 &= \frac{1}{30240}\left(6 S_t^{(4)} - 54 S_t S_t'' - 27 (S_t')^2 + 22 S_t^3\right), \ \ etc
\end{aligned} \qquad (17)$$

- Within our approach we can construct $\mathcal{N}=1,2,3,4$ supersymmetric extension of the Schwarzian basing on the supergroups $OSp(1|2), SU(1,1|1), OSp(3|2), SU(1,1|2)$ and $D(1,2;\alpha)$.
- New set of higher Schwarzians can be constructed
- It is interesting to analyze the supersymmetric versions of the Maxwell algebra and its bosonic extensions by spin 2 fields