Commutative subalgebras in the $\hat{\mathfrak{gl}}_1$ Yangian and Matrix models

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Introduction

- I would like to argue that the study of superintegrability in matrix models and it's relation to the W-representation leads to commutative subalgebras in the affine Yangian $Y(\widehat{\mathfrak{gl}}_1)$

- Outline:
  - Review of the Gaussian matrix model and superintegrability
  - W-operators and $\mathcal{W}_{1+\infty}$ algebra
  - β-deformation, the commutative subalgebras in the Affine Yangian

Matrix models

We deal with integrals of the following kind:

$$Z(p_k) = \int DX \exp\left(-\operatorname{Tr} V(X) + \sum_{k=1}^{\infty} \frac{p_k}{k} \operatorname{Tr} X^K \right)$$

Where $V(X)$ is some potential, and we include $p_k = kt_k$ - generating parameters for all invariant polynomials.

Matrix models

MM posses various features, among those, important for us are:

- Set of Virasoro constraints

$$L_n Z(p_k) = 0, n \geq -1$$

- Integrability - Hirota-like equations

$$D_p \otimes D_{p'} \left(Z(p_k) \otimes Z(p'_k) \right) = 0$$

- Superintegrability:

$$\langle \text{character} \rangle \sim \text{character}$$

- W-representation - evolution in the space of couplings:

$$Z(p_k) = e^W \cdot 1$$

Gaussian matrix model

- Correlators in the GHMM are given by integration over $N \times N$ Hermitian matrices,

$$\langle \ldots \rangle = \int DX \exp\left(-\frac{1}{2}\operatorname{Tr} X^2\right) \ldots$$

  where normalization is included in the measure, i.e we choose $V(X) = -\frac{1}{2}\operatorname{Tr} X^2$.

- The partition function with times included is given by:

$$Z_N^{\text{Gauss}}(p_k) = \int DX \exp\left(-\frac{1}{2}\operatorname{Tr} X^2 + \sum_{k=1}^{\infty} \frac{p_k}{k} \operatorname{Tr} X^K\right)$$

Virasoro constrains

The partition function satisfies Virasoro constraints:

$$L_n Z_N^{\text{Gauss}}(p_k) = 0, n \geq -1$$

$$\left(-(n+2)\frac{\partial}{\partial p_{n+2}} + \sum (k+n) p_k \frac{\partial}{\partial p_{k+n}} + \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} + \right.$$
$$\left. +2Nn\frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + p_1^2 N \delta_{n-1} \right) Z_N^{\text{Gauss}}(p_k) = 0$$

In the Gaussian case the formal series solution of these equations is unique.

Integrability of the GHMM

The partition function enjoys a determinant representation:

$$Z_N^{(\mu)}(p_k) = \det_{1\leq i,j\leq N} M_{i+j-2} = \det_{1\leq i,j\leq N} \left[\left(\frac{\partial}{\partial p_1} \right)^{i+j-2} M_0 \right]$$

$$M_i := \int dx \mu(x) x^i \exp\left(\sum_k \frac{p_k}{k} x^k \right)$$

This guarantees that the partition function is a Toda chain τ-function:

$$Z_N^{(\mu)} \frac{\partial^2 Z_N^{(\mu)}}{\partial p_1^2} - \left(\frac{\partial Z_N^{(\mu)}}{\partial p_1} \right)^2 = Z_{N+1}^{(\mu)} Z_{N-1}^{(\mu)}$$

Superintegrabiltiy of the GHMM

- Computing correlators by Wicks theorem of by solving Virasoro constrains order by order is explicit but still complicated.
- Instead one has a general formula for a specific basis in the space of "observables":

$$\left\langle \mathrm{Schur}_R\left(\mathrm{Tr}\, X^k\right)\right\rangle = \frac{\mathrm{Schur}_R\left\{N\right\}}{\mathrm{Schur}_R\left\{\delta_{k,1}\right\}}\, \mathrm{Schur}_R\left\{\delta_{k,2}\right\}$$

Where Schur_R are Schur functions, for example

$$\mathrm{Schur}_{[2]}(p) = \frac{p_2}{2} + \frac{p_1^2}{2}\,,\; p_k = \mathrm{Tr}\, X^k$$

The r.h.s are Schur functions evaluated at special points

$$\frac{\mathrm{Schur}_R\left\{p_k = N\right\}}{\mathrm{Schur}_R\left\{p_k = \delta_{k,1}\right\}} = \prod_{(i,j)\in R}\left(N + j - i\right)$$

Superintegrability

- One could think that this is a coincidence and a property of simple Gaussian integration.
- However the property holds for many different potentials and deformations:
  - Non-gaussian models, such as higher monomial or logarithmic potentials
  - Potentials with "external" matrices
  - "Non-matrix" deformations of the measure such as the β or (q, t)-deformation

Superintegrability

- Let us give an example and introduce the β-deformation
- Recall that after integration over angular variables we have:

$$Z(p_k) = \int \prod dx_i \Delta^2(x) \exp\left(-\sum_i V(x_i)\right)$$

The β-deformation amounts to

$$\Delta^2(x) \to \Delta^{2\beta}(x)$$

- Superintegrability for a logarithmic potential with β-deformation looks like

$$\int \prod dx_i \Delta(x)^{2\beta} x_i^u (1-x_i)^v \, \mathrm{Jack}_R(x) =$$

$$= \frac{\mathrm{Jack}_R(N)\, \mathrm{Jack}_R(\beta^{-1}u + N + \beta^{-1} - 1)}{\mathrm{Jack}_R(\beta^{-1}(u+v+2) + 2N - 2)}$$

W-representation

- The W-representation can be derived directly from the Virasoro constraints
- Sum up the Virasoro constrainst:

$$0 = \sum_n p_{n+2} L_n Z_N^{\text{Gauss}}(p_k) = \left(\sum_{k=1}^{\infty} k p_k \frac{\partial}{\partial p_k} - W_{-2} \right) Z_N^{\text{Gauss}}(p_k)$$

- Equations is solved by an exponential, called the W-representation

$$Z_N^{\text{Gauss}}(p_k) = e^{\frac{W_{-2}}{2}} \cdot 1$$

- The operator W_{-2} is explicit but somewhat lengthy.
- Such representation is also known for τ-functions

W-representation vs superintegrability

- Partition function admits a character expansion ($S_R = \mathrm{Schur}_R$):

$$\exp\left(\sum \frac{p_k \operatorname{Tr} X^K}{p_k}\right) = \sum_R S_R(p_k) S_R(\operatorname{Tr} X^k)$$

$$\Downarrow$$

$$Z(p_k) = \sum_R \left\langle S_R(\operatorname{Tr} X^k)\right\rangle S_R(p_k) = \sum_R \prod_{(i,j)\in R} (N+j-i)\, S_R\{\delta_{k,2}\} S_R(p_k)$$

- Can superintegrability can be deduced from Virasoro equations?
- A positive answer is provided using the W-representation. The key observation is:

$$W_{-2} S_R = \sum_{Q:|Q|=|R|+2} \left(\prod_{(i,j)\in Q/R} (N+j-i)\right) \left\langle p_2 S_R \middle| S_Q \right\rangle$$

- In general we should study operators with a proper action on Schur functions. The algebra of such operator is exactly the $W_{1+\infty}$ algebra.

- Before the algebra itself let's construct the following family of operators. Start with:

$$W_0 = \sum_{a,b} p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + p_a p_b \frac{\partial}{\partial p_{a+b}} + N \sum p_a \frac{\partial}{\partial p_a}\,, \quad E_0 = p_1$$

- Note that:

$$W_0 S_R \sim \left(\sum_{(i,j)\in R} (N + j - i) \right) S_R$$

- Construct $E_m \sim [\underbrace{W_0, [W_0, \ldots,}_{m} E_0]]$ and

$$H_k^m \sim [\underbrace{E_{m+1}, E_{m+1}, \ldots,}_{k} E_m]$$

- The first claim is that constructed operators for fixed m commute:

$$[H^m_{k_1}, H^m_{k_2}] = 0$$

- The second claim is that their action on characters is given by:

$$H^m_k S_R = \sum_Q \left(\prod_{(i,j)\in Q/R} (N+j-i)^m \right) \left\langle p_k S_R \middle| S_Q \right\rangle, |Q| = |R| + k$$

In particular:

$$W_{-2} = H^1_1$$

- One can generate the following partition functions (hypergeometric τ-functions)

$$Z^m_N(\bar{p}, p) = \exp\left(\sum_k \frac{\bar{p}_k H^m_k}{k} \right) \cdot 1 = \sum_R \left(\prod_{(i,j)R} (N+j-i)^m \right) S_R(\bar{p}) S_R(p)$$

$W_{1+\infty}$ algebra

- The $W_{1+\infty}$ algebra is defined as a deformation of the 2d diffeomorphism group, or the central extension of the algebra of differential operators on the circle $z^m \hat{D}^n$, where $D = z\frac{d}{dz}$:

$$\left[W_n\Big(P(\hat{D})\Big), W_m(Q(\hat{D})\Big)\right] = W_{n+m}\Big(P(\hat{D}+m)Q(\hat{D}) - P(\hat{D})Q(\hat{D}+n)\Big)+$$
$$+\, c\Psi\left(W_n\Big(P(\hat{D})\Big), W_m\Big(Q(\hat{D})\Big)\right)$$

- In terms of these "one-body" operators the commutativity looks especially simple:

$$\hat{H}_k^m = W\left((zD^m)^k\right)$$

- We can obtain the time-variable/bosonic representation of the algebra via an explicit second quantization:

$$W\left(z^n G(\hat{D})\right) = \oint \frac{dz}{2\pi i} z^n \lim_{w\to z} G(\hat{D}_w)\left(\frac{1}{z-w} : e^{\phi(z)-\phi(w)} : - \frac{1}{z-w}\right)$$

where the scalar field is defined as

$$\phi(z) = \sum_{k\geq 1}\left(\frac{\hat{a}_k^\dagger}{k} z^{-k} - z^k \hat{a}_k\right) + \hat{a}_0 + \log(z)\hat{a}_0^\dagger$$

$$\hat{a}_n^\dagger = -p_n, \quad \hat{a}_n = -\frac{\partial}{\partial p_n}$$

- This, along with the explicit iterative commutator formulas from above allows to rather efficiently calculate these operators for practical needs

- We have seen that commutative subalgebras in the $W_{1+\infty}$ algebra are in some sense responsible for matrix model superintegebility
- Is this commutativity a feature of representations? Can we prove it using a set of generators and relations?
- We can answer the last question even for more general operators corressponding to the β-deformation

β-deformation of operators

- In terms of iterative formulas for W-operators the deformation is simple:

$$\hat{W}_0 := \frac{1}{2}\sum_{a,b=1}\left(abp_{a+b}\frac{\partial^2}{\partial p_a \partial p_b} + \beta(a+b)p_a p_b \frac{\partial}{\partial p_{a+b}}\right) + \\ + \beta N_\beta \sum_{a=1} a p_a \frac{\partial}{\partial p_a} + \frac{\beta N_\beta^3}{6} + \frac{1-\beta}{2}\sum_a a(a-1)p_a \frac{\partial}{\partial p_a}$$

where $N_\beta = N + (\beta-1)/2\beta$

- Constructed operators now act on Jack polynomials. For example, relevant for the β-deformed GHMM:

$$H_1^{1,\beta} J_R = \sum_{Q:|Q|=|R|+2}\left(\prod_{(i,j)\in Q/R}(\beta N + j - i\beta)\right)\left\langle p_2 J_R \middle| J_Q\right\rangle$$

Affine Yangian

- However, now the commutation relations between generic operators are more complicated. In particular the is no one body representation
- The relevant algebra is the affine Yangian $Y(\hat{\mathfrak{gl}}_1)$ (for reduced set of parameters)
- This algebra is defined [A. Tsymbaliuk (2017), T. Prochazka (2016)] by a set of generators and relations. Ψ_i, F_i, E_i, $i \in \mathbb{Z}_{\geq 0}$:

$$[\hat{\Psi}_j, \hat{\Psi}_k] = 0$$

$$[\hat{E}_j, \hat{F}_k] = \hat{\Psi}_{j+k}$$

$$[\hat{\Psi}_0, \hat{E}_j] = 0, \qquad [\hat{\Psi}_0, \hat{F}_j] = 0$$

$$[\hat{\Psi}_1, \hat{E}_j] = 0, \qquad [\hat{\Psi}_1, \hat{F}_j] = 0$$

$$[\hat{\Psi}_2, \hat{E}_j] = 2\hat{E}_j, \qquad [\hat{\Psi}_2, \hat{F}_j] = -2\hat{F}_j$$

- Quadratic relations:

$$[\hat{E}_{j+3}, \hat{E}_k] - 3[\hat{E}_{j+2}, \hat{E}_{k+1}] + 3[\hat{E}_{j+1}, \hat{E}_{k+2}] - [\hat{E}_j, \hat{E}_{k+3}] - \\ - [\hat{E}_{j+1}, \hat{E}_k] + [\hat{E}_j, \hat{E}_{k+1}] = \Big(\sigma_3\{\hat{E}_j, \hat{E}_k\} - \sigma_2[\hat{E}_{j+1}, \hat{E}_k] + \sigma_2[\hat{E}_j, \hat{E}_{k+1}]\Big)$$

$$[\hat{\Psi}_{j+3}, \hat{E}_k] - 3[\hat{\Psi}_{j+2}, \hat{E}_{k+1}] + 3[\hat{\Psi}_{j+1}, \hat{E}_{k+2}] - [\hat{\Psi}_j, \hat{E}_{k+3}] - \\ - [\hat{\Psi}_{j+1}, \hat{E}_k] + [\hat{\Psi}_j, \hat{E}_{k+1}] = \dots$$

- Cubic (the Serre relations)

$$\mathrm{Sym}_{i,j,k}[\hat{E}_i, [\hat{E}_j, \hat{E}_{k+1}]] = 0$$

- Similar relatios for F
- The affine Yangian is a 2-parametric family. β deformation corresponds to $\sigma_2 = -1 - \beta(\beta - 1)$, $\sigma_3 = -\beta(\beta - 1)$

- We can prove that commutativity of subalgebras:

$$H_k^m = \mathrm{ad}_{E_{m+1}}^{k+1} E_m$$

  is indeed not just a feature of representation.

- Note, that the proof only uses Serre relations (which are independent of parameters) and Jacobi identities, so it is true for generic parameters in the Yangian

- Everything works for the F counterparts

Concluding remarks

- Superintegrability of matrix models it closely related with commutative subalgebras in the $W_{1+\infty}$ and $Y(\hat{\mathfrak{gl}}_1)$ algebras
- We gave an explicit description of these algebras in terms of iterative commutators, their respective one body operators and their action on Schur/Jack functions
- Commutative subalgebras should correspond to integrable systems. Indeed one can represent each subalgebra as many-body operators. More in Andrei Mironov's lectures.

Further directions:

- I omitted the so-called rational rays, with one-body form: $\left(z^{\pm q} G(\hat{D})\right)^k$. These are more commutative subalgebras in $W_{1+\infty}$. It is unclear how the uplift them to the Yangian
- Matrix modes for all subalgebras. For the first family H_k^1 they are known as WLZZ matrix models.