Trebling of Kähler parameter, Klein bottles and constant maps in the refined Chern-Simons dualities

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- M.Avetisyan and R.Mkrtchyan, Two-fold refinement of non simply laced Chern-Simons theories, Journal of Geometry and Physics 191 (2023) 104907, arXiv:2302.14319,
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- MA, RM, in preparation

Partition function of Chern-Simons on S^3

Partition function of CS on S^3 (Witten, 1989):

$$Z(k) = Vol(Q^\vee)^{-1}(k+h^\vee)^{-\frac{r}{2}} \prod_{\alpha_+} 2\sin\pi\frac{(\alpha,\rho)}{k+h^\vee} \qquad (1)$$

$$Z(0) = 1$$

$Vol(Q^\vee)$ is the volume of the fundamental domain of the coroot lattice $Q^\vee$, the integer k is the CS coupling constant, $h^\vee$ is the dual Coxeter number of the algebra, r is the rank of the algebra, the product is taken over all positive roots α_+.
In an arbitrary normalization of the scalar product:

$$Z(\kappa) = Vol(Q^\vee)^{-1}(\delta)^{-\frac{r}{2}} \prod_{\alpha_+} 2\sin\pi\frac{(\alpha,\rho)}{\delta}$$

where k is now replaced by κ, $h^\vee$ by t, and $\delta = \kappa + t$. In this form the r.h.s. is invariant w.r.t. the simultaneous rescaling of the scalar product, κ, and t (and hence δ). In the minimal normalization, by definition, they accept their usual values in (1).

Equality $Z(0) = 1$ is the Kac-Peterson (1984) identity, expressing the determinant of symmetrized Cartan matrix in terms of product of sines over positive roots:

$$Vol(Q^\vee) = (\det(\alpha_i^\vee, \alpha_j^\vee))^{1/2}$$

$$\alpha_i^\vee = \alpha_i \frac{2}{(\alpha_i, \alpha_i)}, \ \ i = 1, ..., r$$

$$Vol(Q^\vee) = t^{-\frac{r}{2}} \prod_{\alpha_+} 2 \sin \pi \frac{(\alpha, \rho)}{t}$$

For A_{N-1} algebras this equality can be easily proved with the use of the following well-known identity, valid at an arbitrary positive integer N:

$$N = \prod_{k=1}^{N-1} 2 \sin \pi \frac{k}{N}$$

It can be checked for all other root systems, too.

We suggested (Avetisyan, RM, 2021) the following expression for the partition function of the refined CS theory with an arbitrary simple gauge group:

$$Z(\kappa, y) = Vol(Q^\vee)^{-1}\delta^{-\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} 2\sin\pi\frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{\delta}$$

Now $\delta = \kappa + yt$, y is the refinement parameter, which we consider to be a positive integer at this stage.
Key identity $Z(0, y) = 1$ still holds, which is ensured by the following generalization of the Kac-Peterson formula for the same object $Vol(Q^\vee)$:

$$Vol(Q^\vee) = (ty)^{-\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} 2\sin\pi\frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{ty}$$

Initial Macdonald's deformation of simple Lie algebras involves one parameter of deformation for each length of the roots of the algebra. So for an B, C, F, G algebras it can be two deformation (i.e. refinement) parameters. We suggested (MA, RM 2023) the two-fold refinement of the partition function of CS theory on the sphere. It is based on the following two-fold refinement of Kac-Peterson formula:

$$Vol(Q^\vee) = (\tilde{k})^{-\frac{r}{2}} \prod_{\alpha_+} \prod_{m=0}^{k_{\nu_\alpha}-1} 2\sin\pi \frac{k_s(\rho_s,\alpha)+k_l(\rho_l,\alpha)-m(\alpha,\alpha)/2}{\tilde{k}}$$

$$\tilde{k} = k_s(\rho_s,\theta)+k_l(\rho_l,\theta)+k_l\frac{(\theta,\theta)}{2}$$

with θ the maximal root, k_s, k_l are integer refinement parameters for short and long roots, respectively.
Corresponding two-fold refined partition function of CS theory on the sphere is:

$$Z(\kappa,k_s,k_l) = Vol(Q^\vee)^{-1}\delta^{-\frac{r}{2}} \prod_{\alpha_+} \prod_{m=0}^{k_{\nu_\alpha}-1} \sin\pi \frac{k_s(\rho_s,\alpha)+k_l(\rho_l,\alpha)-m(\alpha,\alpha)/2}{\delta}$$

with $\delta = \kappa + \tilde{k}$.
Again, $Z(0,k_s,k_l) = 1$, due to generalized KP identity.

These formulae lead to:

- the duality of refined CS theories (currently carried out for single-refined theories, only, due to complexity of calculations).
- the correct definition of constant maps contribution, and to the trebling of Kähler parameter, due to quantum shifts.
- the description of the conifold singularity by refined group volume formulae.

Transformation of partition function into general values of arguments:

$$Z(\kappa, y) = \left(\frac{ty}{\delta}\right)^{\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} \frac{\sin \pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{\delta}}{\sin \pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{ty}}$$

$$\frac{\sin \pi z}{\pi z} = \frac{1}{\Gamma(1+z)\Gamma(1-z)}$$

and make use of the integral representation of (the logarithm of) the Γ function:

$$\ln \Gamma(1+z) = \int_0^\infty dx \frac{e^{-zx} + z(1-e^{-x}) - 1}{x(e^x - 1)}$$

$$\ln Z = -\frac{1}{4}\int_{R_+} \frac{dx}{x}\,\frac{\sinh\left(x(ty-\delta)\right)}{\sinh\left(xty\right)\sinh\left(x\delta\right)} F_X(2x,y)$$

$$F_X(x,y) = r + \sum_{m=0}^{y-1}\sum_{\alpha_+}\left(e^{x(y(\alpha,\rho)-m(\alpha,\alpha)/2)} + e^{-x(y(\alpha,\rho)-m(\alpha,\alpha)/2)}\right)$$

For example, for the simply-laced algebras (ADE) one have (Krefl, Schwartz, 2013; Avetisyan, RM, 2021)

$$F(x,y) = \frac{\sinh(x\frac{\alpha-2ty}{4})}{\sinh(x\frac{\alpha}{4})}\,\frac{\sinh(xy\frac{\beta-2t}{4})}{\sinh(xy\frac{\beta}{4})}\,\frac{\sinh(xy\frac{\gamma-2t}{4})}{\sinh(xy\frac{\gamma}{4})}$$

where α, β, γ are numbers, Vogel's parameters, characterizing gauge group. At $y = 1$ this coincides with (B.Westbury, 2004).
The advantage of this representation is that it allows one to express the partition function in terms of multiple sine functions.

Simplest example: partition function of refined SU(N) CS on S^3:

$$Z_A(N, y, \delta) = \sqrt{\frac{\delta}{yN}}\frac{S_3(1+yN|1,y,\delta)}{S_3(y|1,y,\delta)} ,$$

This expression includes contribution of Riemann surfaces (S_3 of numerator), and constant maps (denominator S_3, independent of N).

$$\ln S_r(z|\underline{\omega}) = (-1)^r \left(\frac{1}{2}\oint \frac{dx}{x}\frac{e^{zx}}{\prod_{k=1}^r (e^{\omega_i x}-1)} + \int_{R_+}\frac{dx}{x}\frac{e^{zx}}{\prod_{k=1}^r (e^{\omega_i x}-1)} \right) .$$

The expression for $SO(N)$ algebras of type D (i.e., even N) can be written in terms of multiple sine functions as (Krefl. Schwartz, 2013; Krefl, RM, 2015):

$$Z_D = \frac{1}{\sqrt{2}} \frac{S_3(1+ay|1,2y,\delta)}{S_3(y|1,2y,\delta)\, S_2(2-y+ya|2,2\delta)},$$

with $a = N - 1$

For C type algebras partition function is derived to be (Avetisyan, RM, 2022):

$$Z_C = \frac{S_3(1+ya|1,2y,2\delta)}{S_3(y|1,2y,2\delta)} \frac{S_2(2-y+ya|2,4\delta)}{S_2(2-y+ya|2,2\delta)},$$

with $a = N+1$. The analytic derivation of the above Z_C is far more involved than the previous cases. We numerically confirm our analytic derivation.
Finally, for the remaining case of $SO(N)$ group with odd N, i.e., the B series, we have (Avetisyan, RM 2022):

$$Z_B = \frac{1}{2} \frac{S_3(1+ay|1,2y,\delta)}{S_3(1|1,2y,\delta)} \frac{1}{S_2(y(a+1)/2|y,\delta)} \times \frac{S_3(y+ay|1,2y,2\delta)}{S_3(1+ay|1,2y,2\delta)} \frac{S_3(1|1,2y,2\delta)}{S_3(1+y|1,2y,2\delta)},$$

with $a = N-1$. As for the previous cases, we numerically verified this result.

The expressions for D and C partition functions most suitable for a topological string interpretation can be obtained from the initial expressions by the multiple sines manipulations:

$$Z_D = \frac{1}{\sqrt{2}} \frac{\sqrt{S_3(1+ya|1,y,\delta)}}{\sqrt{S_3(y|1,y,\delta)}} \times$$

$$\frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(y|1,2y,\delta)} \frac{S_3(2+ya|1,2y,\delta) S_2(1+ya|2y,\delta)}{\sqrt{S_3(1+ya|1,y,\delta)}\sqrt{S_2(2-y+ya|2,\delta)}} \times$$

$$\frac{\sqrt{S_2(2-y+ya|2,\delta)}}{S_2(2-y+ya|2,2\delta)},$$

$$Z_C = \frac{\sqrt{S_3(1+ya|1,y,\bar{\delta})}}{\sqrt{S_3(y|1,y,\bar{\delta})}} \times$$

$$\frac{\sqrt{S_3(y|1,y,\bar{\delta})}}{S_3(y|1,2y,\bar{\delta})} \frac{S_3(2+ya|1,2y,\bar{\delta}) S_2(1+ya|2y,\bar{\delta})}{\sqrt{S_3(1+ya|1,y,\bar{\delta})}\sqrt{S_2(2-y+ya|2,\bar{\delta})}} \times$$

$$\frac{S_2(2-y+ya|2,2\bar{\delta})}{\sqrt{S_2(2-y+ya|2,\bar{\delta})}},$$

where in the C case we introduced $\bar{\delta} := 2\delta$.

The similar form of the partition function for B:

$$Z_B = \frac{1}{2}\frac{\sqrt{S_3(1+ya|1,y,\delta)}}{\sqrt{S_3(y|1,y,\delta)}} \times$$
$$\frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(1|1,2y,\delta)}\frac{S_3(1|1,2y,2\delta)}{S_3(1+y|1,2y,2\delta)} \times$$
$$\frac{\sqrt{S_3(1+ya+\delta|1,2y,2\delta)}}{\sqrt{S_3(1+ya|1,2y,2\delta)}}\frac{\sqrt{S_3(y+ay+1|1,2y,2\delta)}}{\sqrt{S_3(1+y+ay+\delta|1,2y,2\delta)}}.$$

Gopakumar-Vafa type expansions (Avetisyan, RM, 2022) are:

$$\mathcal{F}_A = \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{\frac{g_s}{2}n(1+2ay-y)}}{n(e^{\frac{g_s}{2}n} - e^{-\frac{g_s}{2}n})(e^{\frac{g_s}{2}ny} - e^{-\frac{g_s}{2}ny})} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{e^{\tau_0 n}}{n(q^n - q^{-n})(t^n - t^{-n})},$$

$$q = e^{\frac{g_s}{2}}, \quad t = e^{\frac{g_s}{2}y}, \quad \tau_0 = \frac{g_s}{2}(1+2ay-y) = g_s ay + \frac{g_s}{2}(1-y),$$

the non-orientable surfaces contribution for D type reads

$$\sum_{n=1,3,\dots} \frac{1}{n}\frac{e^{\frac{1}{2}\tau_1 n}}{(q^n - q^{-n})},$$

$$\tau_1 = g_s(1+ay-y),$$

and the same with opposite overall sign for the C case. The remaining, coinciding, terms in the D and C partition functions (2), (2) can be rewritten as

$$\sum \frac{1}{2n}\frac{\frac{q^n}{t^n} - \frac{t^n}{q^n}}{(q^{2n} - q^{-2n})(t^n + t^{-n})} e^{\tau_2 n}$$

$$\tau_2 = (1+2ay-2y)\frac{g_s}{2}$$

This expression is even over g_s, provided τ_2 is invariant, and can therefore be interpreted as a contribution of a Klein bottle with handles, since the Euler characteristics of such surfaces is even. Note that in the non-refined case there is no such contribution, accordingly, this expression becomes zero at $y = 1$.

In summary, we observe that the single Kähler parameter $\tau = ag_s$ of the non-refined case appears to treble into three parameters after refinement, i.e.,

$$\begin{aligned}\tau_0 = (1 + 2ay - y)\frac{g_s}{2} = y\tau + \frac{1}{2}(1 - y)g_s \\ \tau_1 = (1 + ay - y)g_s = y\tau + (1 - y)g_s \\ \tau_2 = (1 + 2ay - 2y)\frac{g_s}{2} = y\tau + \frac{1}{2}(1 - 2y)g_s \\ \tau = ag_s\end{aligned}$$

for orientable, non-orientable with one cross-cup (i.e. odd Euler characteristics), and non-orientable with two-crosscups (Klein bottles, even Euler characeristics) surfaces, respectively.

Independent confirmation of these conclusions should come from microscopic theory of refined strings, which is absent (see C. Angelantonj, I. Antoniadis, I. Florakis and H. Jiang, Refined topological amplitudes from the Ω-background in string theory, (2022)) For the B type theory we have the same refined A type terms, which correspond to the contributions of Riemannian surfaces, i.e., orientable surfaces with handles. The GV type expansion of the remaining terms (except constant maps) is:

$$\sum_{n=1,3,\dots} \frac{1}{n} \frac{e^{\frac{1}{2}\tau_1^B n}}{(q^{n/2} - q^{-n/2})(t^{n/2} + t^{-n/2})},$$

$$\tau_1^B = g_s(ya - y\frac{1}{2} + \frac{1}{2}) = y\tau + \frac{1}{2}(1-y)g_s.$$

This is the deformed version of the one-cross-cup term of the non-refined *SO* theory. It is odd w.r.t. the string coupling g_s, provided τ_1^B is unchanged. Note that the deformation τ_1^B is different from τ_1 in the D (and C) cases. We conclude that in the B case there is no Klein bottle contribution, as in the non-refined case. The initial Kähler parameter τ is shifted for Riemannian surfaces and for unorientable one cross-cup surfaces as follows:

$$\tau_0^B = (1 + 2ay - y)\frac{g_s}{2} = y\tau + \frac{1}{2}(1-y)g_s,$$

$$\tau_1^B = g_s(ya - y\frac{1}{2} + \frac{1}{2}) = y\tau + \frac{1}{2}(1-y)g_s,$$

$$\tau = ag_s.$$

We conclude that for the B case the shifts are identical, i.e., $\tau_0^B = \tau_1^B$. Taking into account the absence of Klein bottles contribution, we see that refined B case is very close to the unrefined D case.

Refined constant maps

$$\frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(y|1,2y,\delta)}$$

The perturbative poles contribution in this expression is as follows:

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(\frac{t}{q})^n}{n(q^n-q^{-n})(t^n+t^{-n})}$$

This term is neither even, nor odd over the coupling constant g_s ($g_s \to -g_s$ is equivalent to $q \to 1/q, t \to 1/t$). It remains non-zero in the limit $y = 1$ ($q = t$):

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(q^{2n}-q^{-2n})}$$

However, the initial expression in the same limit $y = 1$ is

$$\frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(y|1,2y,\delta)} = \frac{1}{\sqrt{S_2(1|2,\delta)}} = 2^{-\frac{1}{4}}$$

so its expansion should be zero. This happens since perturbative and non-perturbative terms cancel, due to modular properties of functions.

To handle this ambiguity, we suggest (Avetisyan, RM, 2022) to rewrite all terms corresponding to the constant maps in a "modularity-neutral" form, i.e. as follows:

$$S_r(z|w) = \sqrt{S_r^2} = \sqrt{S_r(z|w)S_r(|w|-z|w)^{(-1)^{r+1}}}$$

The refined constant maps term in above, corresponding to the Riemann surfaces (i.e. orientable worldsheets), in GV form will contribute by the following expression:

$$\ln S_3(y|1,y,\delta) = \ln\sqrt{S_3(y|1,y,\delta)S_3(1+\delta|1,y,\delta)} \sim$$
$$-\frac{1}{2}\sum\frac{1}{n}\frac{\frac{q^n}{t^n}+\frac{t^n}{q^n}}{(q^n-q^{-n})(t^n-t^{-n})}$$
$$q = e^{\frac{g_s}{2}}, \quad t = e^{\frac{g_s y}{2}}$$

This expression has the same limit in unrefined case at $y = 1$, however, it is explicitly even w.r.t. the string coupling g_s.

Without our recipe, the initial constant maps term $S_3(y|1,y,\delta)$ of refined A theory leads to the expansion below, which, to be even in string coupling, require invariance w.r.t. the $y \to 1/y, g_s \to y g_s$, which exists in the A case, only.

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-ng_s(1-y)/2}}{n\sinh\left(\frac{ng_s}{2}\right)\sinh\left(\frac{nyg_s}{2}\right)},$$

Next, consider the new form of the constant maps term for the refined D theory:

$$\ln \frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(y|1,2y,\delta)} \sim$$

$$\ln \frac{\sqrt{\sqrt{S_3(y|1,y,\delta)S_3(1+\delta|1,y,\delta)}}}{\sqrt{S_3(y|1,2y,\delta)S_3(1+y+\delta|1,2y,\delta)}} \sim$$

$$-\frac{1}{4}\sum \frac{1}{n}\frac{\frac{q^n}{t^n}-\frac{t^n}{q^n}}{(q^n-q^{-n})(t^n+t^{-n})}$$

Remarkably, this expression is even in g_s, which allows us to assume that it corresponds to the constant maps term of a two cross-cup non-orientable worldsheet contribution.
Consider remaining constant maps terms, namely those in B case, two fractions. They are

$$\ln \frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(1|1,2y,\delta)} \sim \sum_{n=1} \frac{\frac{t^n}{q^n}-\frac{q^n}{t^n}}{4n\left(q^n-q^{-n}\right)\left(t^{-n}+t^n\right)}$$

$$\ln \frac{S_3(1|1,2y,2\delta)}{S_3(1+y|1,2y,2\delta)} \sim -\sum_{n=1} \frac{\frac{t^{n/2}}{q^{n/2}}-\frac{q^{n/2}}{t^{n/2}}}{2n\left(q^{n/2}-q^{-\frac{n}{2}}\right)\left(t^{-\frac{n}{2}}+t^{\frac{n}{2}}\right)}$$

so the sum of these two terms is the sum over odd n :

$$\ln \frac{\sqrt{S_3(y|1,y,\delta)}}{S_3(1|1,2y,\delta)} \frac{S_3(1|1,2y,2\delta)}{S_3(1+y|1,2y,2\delta)} \sim$$

$$-\sum_{n=1,3,\ldots} \frac{\frac{t^{n/2}}{q^{n/2}} - \frac{q^{n/2}}{t^{n/2}}}{2n\left(q^{n/2} - q^{-\frac{n}{2}}\right)\left(t^{\frac{n}{2}} + t^{-\frac{n}{2}}\right)}$$

We see, that the answer is even w.r.t. the string coupling, so we assume that it again is coming from Klein bottles, representing their refined constant maps contribution on orientifolded resolved conifold. At $y = 1$ this expression is zero, as it should.

String coupling and Kähler parameter expansions

$$\sum_{n=1}^{\infty} \frac{e^{\tau n}}{n(q^n - q^{-n})(t^n - t^{-n})} \, ,$$

with

$$q = e^{\frac{g_s}{2}} \, , \quad t = e^{\frac{g_s}{2} y} \, , \quad \tau = \frac{g_s}{2}(1 + 2ay - y) \, ,$$

$$\frac{1}{e^x - e^{-x}} = \sum_{n=0}^{\infty} \frac{(1 - 2^{2n-1})}{(2n)!} B_{2n} x^{2n-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{\tau n}}{(q^n - q^{-n})(t^n - t^{-n})} =$$

$$\sum_{g=0}^{\infty} (\frac{g_s}{2})^{2g-2} \sum_{n=1}^{\infty} \left(n^{2g-3} e^{\tau n} \right) \sum_{p=0}^{g} \frac{1 - 2^{2g-2p-1}}{(2g-2p)!} B_{2g-2p} \frac{1 - 2^{2p-1}}{(2p)!} B_{2p} y^{2p-1} =$$

$$\sum_{g=0}^{\infty} F_g^0 (\frac{g_s}{2})^{2g-2}$$

Conifold singularity

$$\sum_{n=1}^{\infty} n^k e^{n\tau} = \frac{(-1)^{k+1} k!}{\tau^{k+1}} - \sum_{m=k+1}^{\infty} \frac{1}{m(m-k-1)!} B_m \tau^{m-k-1}, \quad k > 0$$

$$F^0_{g>1} = \left(\frac{(2g-3)!}{\tau^{2g-2}} - \sum_{k=0}^{\infty} \frac{1}{(2g+2k-2)(2k)!} B_{2g+2k-2} \tau^{2k} \right) \times$$
$$\sum_{p=0}^{g} \frac{1-2^{2g-2p-1}}{(2g-2p)!} B_{2g-2p} \frac{1-2^{2p-1}}{(2p)!} B_{2p} y^{2p-1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{\tau n}}{(q^n - q^{-n})(t^n - t^{-n})} \sim$$
$$\sum_{g=0}^{\infty} \left(\frac{g_s}{2\tau}\right)^{2g-2} (2g-3)! \sum_{p=0}^{g} \frac{1-2^{2g-2p-1}}{(2g-2p)!} B_{2g-2p} \frac{1-2^{2p-1}}{(2p)!} B_{2p} y^{2p-1}$$

Conifold singularity

Natural conifold limit:

$$\tau \to 0, g_s \to 0, \frac{g_s}{\tau} = const$$

Partition function at conifold singularity (differs from D.Krefl and J.Walcher, 2012), (MA, RM, in preparation):

$$\ln Z \sim \int_0^\infty \frac{dx}{x}\left(F_X(\frac{x}{\delta}) - dim(y)\right) - \int_0^\infty \frac{dx}{x}\left(F_X(\frac{x}{ty}) - dim(y)\right)$$

$$\delta = 1/g_s$$

$$\to -\int_0^\infty \frac{dx}{x}\left(F_X(\frac{x}{ty}) - dim(y)\right) \sim \ln(Vol_G(y))$$

This is refined volume of groups, to be studied.

Thanks!