Non-linear spin multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$

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Introduction

- Coupling of dynamical and semi-dynamical multiplets was proposed by S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Phys. Rev. D* **79** (2009) 105015. This idea provided harmonic superfield construction of $\mathcal{N}=4$ extension of Calogero system with the additional spin (isospin) degrees of freedom z^i, $\bar{z}_j$.

- This work was followed by a further study of "spinning" models considering couplings of dynamical and semi-dynamical multiplets (S. Bellucci, S. Krivonos, A. Sutulin, *Phys. Rev. D* **81** (2010) 105026, S. Krivonos, O. Lechtenfeld, A. Sutulin, *Phys. Rev. D* **81** (2010) 085021, E. Ivanov, M. Konyushikhin, A. Smilga, *JHEP* **1005** (2010) 033, E. Ivanov, M. Konyushikhin, *Phys. Rev. D* **82** (2010) 085014, etc). Most of them considered the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ as a semi-dynamical.

- The multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as a semi-dynamical multiplet interacting with the dynamical multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ was considered by S. Fedoruk, E. Ivanov, O. Lechtenfeld, *JHEP* **1206** (2012) 147. They showed that the triplet of spin variables v^{ij} describes a 2 dimensional surface in $\mathbb{R}^3$.

- Recently, we considered the interaction of the semi-dynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ with dynamical mirror multiplets (E. Ivanov, S. S., *Phys. Rev. D* **105** (2022) 086027). We reproduced the model mentioned above, but in terms of mirror superfields.

- We constructed, for the first time, the coupling of the semi-dynamical mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ to the dynamical mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ described by the chiral superfield Z. The corresponding interaction term was constructed as a superpotential in the chiral subspace.

- Our goal is to consider the non-linear mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as a spin multiplet. As a generalization of the linear case the triplet of spin variables $v^{\alpha\beta}$ describes a 2 dimensional surface in S^3.

$\mathcal{N}=4,\ d=1$ supersymmetry

The standard $\mathcal{N}=4,\ d=1$ superalgebra:

$$\left\{Q^i_\alpha, Q^\beta_j\right\} = 2\delta^i_j\delta^\beta_\alpha H. \tag{1}$$

Supercharges Q^i_α carry fundamental indices ($i=1,2$ and $\alpha=1,2$) of the automorphism group $\mathrm{SO}(4) \sim \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$. The superspace is

$$\zeta := \left\{t, \theta^{i\alpha}\right\}, \tag{2}$$

and transforms as

$$\delta\theta^{i\alpha} = \epsilon^{i\alpha}, \qquad \delta t = -\,i\,\epsilon^{i\alpha}\theta_{i\alpha}\,, \qquad \overline{(\theta^{i\alpha})} = -\,\theta_{i\alpha}\,, \qquad \overline{(\epsilon^{i\alpha})} = -\,\epsilon_{i\alpha}\,. \tag{3}$$

The covariant derivatives are

$$D^{i\alpha} = \frac{\partial}{\partial\theta_{i\alpha}} + i\,\theta^{i\alpha}\partial_t\,. \tag{4}$$

The multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is described by a triplet superfield $\mathcal{V}^{ij}$ satisfying

$$D^{(k}_\alpha\mathcal{V}^{ij)} = 0. \tag{5}$$

Harmonic superspace construction

The multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ can be described by a harmonic superfield $\mathcal{V}^{++}$ that lives on the analytic harmonic superspace $\zeta_{(\mathrm{A})}$ (E. Ivanov, O. Lechtenfeld, *JHEP* **0309** (2003) 073). The superfield satisfies the analyticity constraints:

$$D^{+\alpha}\mathcal{V}^{++} = 0, \qquad D^{++}\mathcal{V}^{++} = 0. \tag{6}$$

So called Wess-Zumino (WZ) type Lagrangian is constructed as an analytic superpotential for $\mathcal{V}^{++}$:

$$S_{\mathrm{WZ}} = \int dt\, du\, D^{-\alpha} D^{-}_{\alpha}\, \mathcal{L}^{++}\left(\mathcal{V}^{++}, u^{\pm}_{i}\right), \qquad D^{+\alpha}\mathcal{L}^{++}\left(\mathcal{V}^{++}, u^{\pm}_{i}\right) = 0. \tag{7}$$

In components, it has the following form

$$S_{\mathrm{WZ}} = \int dt \left[D\,\mathcal{U}\left(v\right) + i\dot{v}^{ij}\mathcal{A}_{ij}\left(v\right) + \frac{1}{2}\,\psi^{(i}_{\alpha}\psi^{j)\alpha}\,\mathcal{R}_{ij}\left(v\right)\right]. \tag{8}$$

It does not contain second-order terms in time derivatives, only first-order bosonic terms. By itself this Lagrangian describes a semi-dynamical spin multiplet. The triplet v^{ij} describes semi-dynamical degrees of freedom (or spin variables), while the fermionic fields $\psi^{i\alpha}$ become auxiliary, and the singlet D can be treated as Lagrange multiplier (S. Fedoruk, E. Ivanov, O. Lechtenfeld, *JHEP* **1206** (2012) 147).

Linear mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$

The ordinary $\mathcal{N}=4$ multiplets have their mirror counterparts characterized by the mutual interchange of two $\mathrm{SU}(2)$ groups which form the $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ automorphism group (E. Ivanov, J. Niederle, *Phys. Rev. D* **80** (2009) 065027). Swapping the indices $i,j\leftrightarrow\alpha,\beta$ yields the same $\mathcal{N}=4$, $d=1$ superalgebra:

$$\left\{Q^i_\alpha, Q^\beta_j\right\} = 2\delta^i_j\delta^\beta_\alpha H, \qquad i,j\leftrightarrow\alpha,\beta. \tag{9}$$

Thus the constraints for the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ are written as

$$D^{i(\alpha}V^{\beta\gamma)}=0, \qquad V^{\alpha\beta}=V^{\beta\alpha}, \qquad \overline{(V^{\alpha\beta})}=-\,V_{\alpha\beta}\,. \tag{10}$$

Both multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ are mutually equivalent. We can split the triplet $V^{\alpha\beta}$ into complex and real superfields as

$$V^{12}=-\,Y, \qquad V^{22}=-\,\sqrt{2}\,U, \qquad V^{11}=\sqrt{2}\,\bar{U}. \tag{11}$$

The constraints become

$$D^i\bar{U}=0, \qquad \bar{D}_iU=0, \qquad \sqrt{2}\,D_iY=\bar{D}_i\bar{U}, \qquad \sqrt{2}\,\bar{D}_iY=-\,D_iU. \tag{12}$$

The complex superfield is chiral (S. S., *J. Phys. A* **54** (2021) 035205).

Non-linear mirror multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$

The non-linear mirror multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ is described by a quartet superfield $N^{\alpha A}$, where the capital index A $(A = 1, 2)$ is an external $\mathrm{SU}(2)_{\mathrm{ext.}}$ index. The superfield satisfies the following constraints (E. Ivanov, S. Krivonos, O. Lechtenfeld, *Class. Quant. Grav.* **21** (2004) 1031-1050)

$$N^{\alpha A} N_{\alpha A} = R^2, \qquad N_A^{(\alpha} D_i^{\beta} N^{\gamma) A} = 0. \tag{13}$$

The first constraint specifies coordinates of the sphere S^3 in $\mathbb{R}^4$. The constraints are written covariantly with respect to the external $\mathrm{SU}(2)_{\mathrm{ext.}}$ group and the automorphism group $\mathrm{SO}(4) \sim \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$. The subgroup $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{ext.}}$ corresponds to the $\mathrm{SO}(4)$ isometry group of S^3.

Stereographic coordinates

We introduce the stereographic coordinates $V^{\alpha\beta}$ of S^3 as

$$N^{21}=\frac{R\left[2iR\,V_{12}+\left(R^2-V^2\right)\right]}{\sqrt{2}\,(R^2+V^2)}\,,\qquad N^{11}=\frac{-\sqrt{2}\,iR^2\,V^{11}}{R^2+V^2}\,,$$
$$N^{12}=\frac{R\left[2iR\,V_{12}-\left(R^2-V^2\right)\right]}{\sqrt{2}\,(R^2+V^2)}\,,\qquad N^{22}=\frac{-\sqrt{2}\,iR^2\,V^{22}}{R^2+V^2}\,, \tag{14}$$

where

$$V^2=-\,\frac{1}{2}\,V^{\alpha\beta}V_{\alpha\beta}\,,\qquad \overline{(V^{\alpha\beta})}=-\,V_{\alpha\beta}\,. \tag{15}$$

The relevant metric on the 3-sphere reads

$$ds^2=-\,\frac{R^4\,dv^{\alpha\beta}dv_{\alpha\beta}}{(R^2+v^2)^2}\,,\qquad v^{\alpha\beta}:=V^{\alpha\beta}\,|_{\theta=0}\,,\qquad v^2:=-\,\frac{1}{2}\,v^{\alpha\beta}v_{\alpha\beta}\,. \tag{16}$$

The superfield constraints are rewritten as

$$D_i^{(\gamma}V^{\alpha\beta)} + \frac{i}{R}\,V^{\lambda(\gamma}D_{i\lambda}V^{\alpha\beta)} = 0. \tag{17}$$

One can see that the limit $R \to \infty$ leads to the linear constraints of the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$. The non-linear constraints are invariant under the following rotations and translations on the 3-sphere:

$$\begin{aligned} \delta V^{\alpha\beta} &= a^{\alpha\beta} - \frac{2i}{R}\,a_\lambda^{(\alpha}V^{\beta)\lambda} - \frac{1}{R^2}\,a^{\lambda\mu}V_\lambda^\alpha V_\mu^\beta\,, \\ \delta V^{\alpha\beta} &= 2b_\lambda^{(\alpha}V^{\beta)\lambda}, \qquad \delta D_i^\alpha = b_\lambda^\alpha D_i^\lambda. \end{aligned} \tag{18}$$

The rotations $b^{\alpha\beta}$ correspond to a diagonal subgroup of $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{ext.}}$, while the translations $a^{\alpha\beta}$ represents the external group $\mathrm{SU}(2)_{\mathrm{ext.}}$:

$$\begin{aligned} \delta N^{\alpha A} &= b_\beta^\alpha N^{\beta A} + b_B^A N^{\alpha B}, \qquad \delta D_i^\alpha = b_\beta^\alpha D_i^\beta\,, \\ \delta N^{\alpha A} &= -\,\frac{2i}{R}\,a_B^A\,N^{\alpha B}. \end{aligned} \tag{19}$$

Because we are interested in construction of WZ action, it is enough to present the θ and θ^2 expansions of the superfield. The non-linear solution reads

$$
\begin{aligned}
V^{\alpha\beta} \;=\;& v^{\alpha\beta} - \theta^{k(\alpha}\chi_k^{\beta)} - \frac{i}{R}\,\theta^k_\lambda v^{\lambda(\alpha}\chi_k^{\beta)} - \left[\frac{1}{2}\,\theta^{k(\alpha}\theta_k^{\beta)} + \frac{i}{R}\,\theta^{k(\alpha}\theta_{k\gamma}v^{\beta)\gamma} - \frac{1}{2R^2}\,\theta^{k(\lambda}\theta_k^{\gamma)}v_\lambda^{(\alpha}v_\gamma^{\beta)}\right] C \\
& + \frac{1}{R^2+v^2}\left[iR^2\,\theta^{k(\alpha}\theta_{k\gamma}\dot{v}^{\beta)\gamma} - 2R\,\theta^{k(\alpha}\theta_{k\gamma}\dot{v}^{\beta)\mu}v^\gamma_\mu - R\,\theta^k_\gamma\theta_{k\mu}v^{\gamma\mu}\dot{v}^{\alpha\beta} + i\,\theta^{k\gamma}\theta_{k\lambda}v^\lambda_\mu v_\gamma^{(\alpha}\dot{v}^{\beta)\mu}\right] \\
& + \frac{\theta^{k(\lambda}\theta_k^{\mu)}\,v^{\alpha\beta}}{8\,(R^2+v^2)}\left(\chi^i_\lambda\chi_{i\mu} - \frac{2i}{R}\,v^\gamma_\lambda\,\chi^i_\gamma\chi_{i\mu} - \frac{1}{R^2}\,v^\gamma_\lambda\,v^\delta_\mu\,\chi^i_\gamma\chi_{i\delta}\right) - \frac{1}{8R^2}\,\theta_\lambda^{(i}\theta^{k)\lambda}v^{\alpha\beta}\chi_{i\mu}\chi_k^\mu \\
& - \frac{i}{2R}\,\theta^{k(\alpha}\theta_{k\gamma}\chi^{i\beta)}\chi_i^\gamma + \frac{1}{2R^2}\,\theta^{k(\lambda}\theta_k^{\mu)}\chi^i_\lambda\chi_i^{(\alpha}v_\mu^{\beta)} + \left(\theta^3 \text{ and } \theta^4 \text{ terms}\right), \qquad (20)
\end{aligned}
$$

where

$$
\overline{(v^{\alpha\beta})} = -\,v_{\alpha\beta}, \qquad \overline{(\chi^{k\alpha})} = -\,\chi_{k\alpha}\,, \qquad \overline{(C)} = C. \qquad (21)
$$

The components transform as

$$\begin{aligned}
\delta v^{\alpha\beta} &= \epsilon^{k(\alpha}\chi_k^{\beta)} + \frac{i}{R}\,\epsilon^k_\lambda v^{\lambda(\alpha}\chi_k^{\beta)},\\
\delta\chi_i^\alpha &= \frac{2iR^2\,\epsilon_{i\beta}}{R^2+v^2}\left(\dot v^{\alpha\beta} + \frac{i}{R}\,v^\beta_\gamma \dot v^{\alpha\gamma}\right) - \left(\epsilon_i^\alpha + \frac{i}{R}\,\epsilon_{i\beta}v^{\alpha\beta}\right)C + \left(\epsilon_i^\alpha + \frac{i}{R}\,\epsilon_{i\beta}v^{\alpha\beta}\right)\frac{v_{\lambda\mu}\,\chi^{i\lambda}\chi_i^\mu}{4\,(R^2+v^2)}\\
&\quad + \frac{i}{2R}\,\epsilon^{k\beta}\chi_{i\beta}\chi_k^\alpha - \frac{iR\,\epsilon_{i\beta}}{2\,(R^2+v^2)}\left(\chi^{k\beta} + \frac{i}{R}\,v^\beta_\gamma\chi^{k\gamma}\right)\chi_k^\alpha,\\
\delta C &= -\,i\,\epsilon_{i\alpha}\,\partial_t\left[\frac{R^2}{R^2+v^2}\left(\chi^{i\alpha} + \frac{i}{R}\,\chi^{i\beta}v^\alpha_\beta\right)\right].
\end{aligned} \tag{22}$$

WZ action

The sigma-model type and WZ Lagrangians for the non-linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ were constructed by S. Bellucci, S. Krivonos, *Phys. Rev. D* **74** (2006) 125024. Here, we separately consider WZ Lagrangian and give it in a manifestly $\mathcal{N}=4$ supersymmetric superfield form. Ansatz for the WZ action reads

$$S_{\rm WZ} = \int dt\, d^4\theta \left[\theta^{k(\alpha}\theta^{\beta)}_k\, L_{\alpha\beta}\,(V)\right] = \frac{1}{12}\int dt\, D_{i\lambda} D^{\lambda}_j D^{(i}_{\mu} D^{j)\mu} \left[\theta^{k(\alpha}\theta^{\beta)}_k\, L_{\alpha\beta}\,(V)\right],$$
$$L^{\alpha\beta} = L^{\beta\alpha}, \qquad \overline{(L_{\alpha\beta})} = -\,L^{\alpha\beta}. \tag{23}$$

We impose a quadratic condition,

$$D^{(i}_{\gamma} D^{j)\gamma} L^{\alpha\beta}\,(V) = 0, \tag{24}$$

which is a sufficient condition,

$$\delta S_{\rm WZ} = \frac{1}{6}\int dt\, D_{i\lambda} D^{\lambda}_j D^{(i}_{\mu} D^{j)\mu} \left[\epsilon^{k(\alpha}\theta^{\beta)}_k\, L_{\alpha\beta}\,(V)\right] \quad \Rightarrow \quad \int dt\, \epsilon_{j\alpha} D_{i\beta} D^{(i}_{\mu} D^{j)\mu} L^{\alpha\beta}\,(V) = 0. \tag{25}$$

It leads to

$$\frac{1}{3}\left(D^{(i}_{\gamma} V_{\delta\rho}\right)\left(D^{j)\gamma} V^{\delta\rho}\right)\left[\frac{\partial}{\partial V_{\lambda\mu}} + \frac{V^{\lambda\mu}}{R^2+V^2}\right]\frac{\partial}{\partial V^{\lambda\mu}} L^{\alpha\beta}\,(V) = 0. \tag{26}$$

WZ Langrangian

The triplet function $L^{\alpha\beta}(v)$ satisfies the Laplace-Beltrami equation on S^3:

$$\Delta_{S^3} L^{\alpha\beta}(v) = 0, \qquad \Delta_{S^3} = -\frac{1}{R^4}\left(R^2+v^2\right)^2\left(\partial^{\lambda\mu} + \frac{v^{\lambda\mu}}{R^2+v^2}\right)\partial_{\lambda\mu}\,, \qquad \partial_{\lambda\mu} v^{\alpha\beta} = \delta_\lambda^{(\alpha}\delta_\mu^{\beta)}. \tag{27}$$

In components we obtain the WZ Langrangian

$$\mathcal{L}_{\rm WZ} = C\,\mathcal{U}(v) + i\,\dot v^{\alpha\beta}\mathcal{A}_{\alpha\beta}(v) + \frac{1}{2}\,\chi^{i\alpha}\chi_i^{\beta}\,\mathcal{R}_{\alpha\beta}(v)\,, \tag{28}$$

where

$$\begin{aligned}
\mathcal{U}(v) &= \partial_{\alpha\beta}L^{\alpha\beta} - \frac{2i}{R}\,v_\alpha^\gamma\,\partial_{\beta\gamma}L^{\alpha\beta} - \frac{1}{R^2}\,v_\alpha^\gamma\,v_\beta^\lambda\,\partial_{\gamma\lambda}L^{\alpha\beta},\\
\mathcal{A}_{\alpha\beta}(v) &= \frac{2}{R^2+v^2}\Big(-R^2\,\partial_{\gamma(\alpha}L^\gamma_{\beta)} + 2iR\,v_{\gamma(\alpha}\,\partial_{\beta)\lambda}L^{\gamma\lambda} - iR\,v_{\gamma\lambda}\,\partial_{\alpha\beta}L^{\gamma\lambda} + v_\lambda^\rho\,v_{\gamma(\alpha}\,\partial_{\beta)\rho}L^{\gamma\lambda}\Big),\\
\mathcal{R}_{\alpha\beta}(v) &= \partial_{\mu(\alpha}\,\partial_{\beta)\lambda}L^{\lambda\mu} + \frac{2i}{R}\Big(\partial_{\gamma(\alpha}L^\gamma_{\beta)} - v_\lambda^\gamma\,\partial_{\mu(\alpha}\,\partial_{\beta)\gamma}L^{\lambda\mu}\Big)\\
&\quad - \frac{1}{R^2}\Big(2\,v^{\lambda\mu}\,\partial_{\lambda(\alpha}L_{\beta)\mu} + v_\lambda^\gamma v_\mu^\rho\,\partial_{\rho(\alpha}\,\partial_{\beta)\gamma}L^{\lambda\mu}\Big)\\
&\quad - \frac{1}{2\,(R^2+v^2)}\Big(v^{\lambda\mu}\,\partial_{\lambda\mu}L_{\alpha\beta} + \frac{2i}{R}\,v_\alpha^\gamma\,v^{\lambda\mu}\,\partial_{\lambda\mu}L_{\gamma\beta} - \frac{1}{R^2}\,v_\alpha^\gamma\,v_\beta^\delta\,v^{\lambda\mu}\,\partial_{\lambda\mu}L_{\gamma\delta}\Big).
\end{aligned} \tag{29}$$

The supersymmetry requires the following conditions:

$$\partial^{\gamma}_{(\alpha}\,\mathcal{A}_{\beta)\gamma} = \frac{R^2\,\partial_{\alpha\beta}\,\mathcal{U}}{R^2+v^2}\,, \qquad \mathcal{R}_{\alpha\beta} = \partial_{\alpha\beta}\,\mathcal{U}. \tag{30}$$

Both equations can be rewritten as

$$\mathbf{rot}_{S^3}\,\mathcal{A} = \mathbf{grad}_{S^3}\,\mathcal{U}, \qquad \mathcal{R}_{\alpha\beta} = \nabla_{\alpha\beta}\,\mathcal{U}. \tag{31}$$

One can calculate that the scalar potential $\mathcal{U}$ satisfies the Laplace-Beltrami equation:

$$\mathbf{\Delta}_{S^3}\,\mathcal{U} = 0. \tag{32}$$

The auxiliary field C plays the role of a Lagrange multiplier. Its equation of motion enforces the constraint

$$\mathcal{U}\,(v) \approx 0. \tag{33}$$

It kills one degree of freedom in the triplet $v^{\alpha\beta}$. Consequently, the group $\mathrm{SO}(4)$ reduces to the isometry group of a 2-dimensional surface embedded in S^3.

Dirac brackets

The fermionic fields $\chi^{i\alpha}$ can be eliminated by their equations of motion. Then pass to the Hamiltonian system with $\lambda^{\alpha\beta}$ and C treated as Lagrange multipliers:

$$H = \lambda^{\alpha\beta}\pi_{\alpha\beta} - C\,\mathcal{U}. \tag{34}$$

The second class Hamiltonian constraints of the system are then given by

$$\pi_{\alpha\beta} = p_{\alpha\beta} - i\,\mathcal{A}_{\alpha\beta} \approx 0, \qquad \mathcal{U} \approx 0. \tag{35}$$

The last constraint appears as a secondary one from the primary constraint $p_C \approx 0$. Define a matrix formed by Poisson brackets of these constraints:

$$M = \begin{pmatrix} \{\pi_{\alpha\beta},\pi_{\gamma\delta}\}_{\mathrm{PB}} & \{\pi_{\alpha\beta},\mathcal{U}\}_{\mathrm{PB}} \\ \{\mathcal{U},\pi_{\gamma\delta}\}_{\mathrm{PB}} & 0 \end{pmatrix}. \tag{36}$$

Following the Dirac procedure we calculate the inverse matrix M^{-1} and find Dirac brackets as

$$\left\{v^{\alpha\beta}, v_{\lambda\mu}\right\}_{\mathrm{DB}} = \frac{i\,\delta^{(\alpha}_{(\lambda}\partial^{\beta)}_{\mu)}\mathcal{U}}{\partial_{\gamma\delta}\mathcal{U}\,\partial^{\gamma\delta}\mathcal{U}}\left(1+\frac{v^2}{R^2}\right), \qquad \partial_{\gamma\delta}\mathcal{U}\,\partial^{\gamma\delta}\mathcal{U} \neq 0. \tag{37}$$

Monopole

The monopole solution (or a fuzzy sphere S^2) for the linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ was constructed in the analytic harmonic superspace (E. Ivanov, O. Lechtenfeld, *JHEP* **0309** (2003) 073). Counterpart of this solution for the non-linear mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ is given by

$$L_{\alpha\beta}\left(V\right) = \frac{2k_{\alpha\beta}V^2 + k^{\lambda\mu}V_{\lambda\mu}V_{\alpha\beta}}{2|V|\left(2|k||V| - k^{\lambda\mu}V_{\lambda\mu}\right)}, \qquad k^2 = -\frac{1}{2}\,k^{\alpha\beta}k_{\alpha\beta}\,, \tag{38}$$

where $k^{\alpha\beta}$ is a constant vector. The sphere Lagrangian reads

$$\mathcal{L}_{\rm sphere} = C\,\mathcal{U}\left(v\right) + i\,\dot{v}^{\alpha\beta}\mathcal{A}_{\alpha\beta}\left(v\right) + \frac{1}{2}\,\chi^{i\alpha}\chi_i^{\beta}\,\mathcal{R}_{\alpha\beta}\left(v\right), \tag{39}$$

where

$$\mathcal{U}\left(v\right) = -\frac{1}{2|v|}\left(1 - \frac{v^2}{R^2}\right), \qquad \mathcal{A}_{\alpha\beta} = -\frac{k^{\gamma}_{(\alpha}v_{\beta)\gamma}}{|v|\left(2|k||v| - k^{\lambda\mu}v_{\lambda\mu}\right)} - \frac{iR\,v_{\alpha\beta}}{|v|\left(R^2 + v^2\right)}\,. \tag{40}$$

Sphere S^2

From the constraint $\mathcal{U} \approx 0$ we derive the equation of the sphere S^2:

$$v^2 \approx R^2. \tag{41}$$

For clarification, it is convenient to pass to coordinates in $\mathbb{R}^4$ satisfying

$$n^{\alpha A} n_{\alpha A} = (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = R^2, \qquad n^{\alpha A} := N^{\alpha A}\,|_{\theta=0}\,. \tag{42}$$

The constraint becomes

$$\mathcal{U}\,(x_1, x_2, x_3, x_4) = \frac{x_3}{R\left[(x_1)^2 + (x_2)^2 + (x_4)^2\right]^{\frac{1}{2}}} \approx 0. \tag{43}$$

Thus, the sphere S^2 is obtained as a result of the section of S^3 by the plane $x_3 = 0$.

We can add to $\mathcal{L}_{\text{sphere}}$ the Fayet-Iliopoulos (FI) term

$$\mathcal{L}_{\text{FI}} = \frac{C}{2r}\left(1 - \frac{r^2}{R^2}\right), \qquad r = \text{const.}, \tag{44}$$

and modify the constraint as

$$\mathcal{U}(v) = \frac{1}{2r}\left(1 - \frac{r^2}{R^2}\right) - \frac{1}{2|v|}\left(1 - \frac{v^2}{R^2}\right) \approx 0. \tag{45}$$

Here r is a radius of the sphere S^2 which is a parameter independent of R. In the limit $R \to \infty$ we obtain the sphere S^2 embedded in $\mathbb{R}^3$:

$$\mathcal{U}(v) = \frac{1}{2r} - \frac{1}{2|v|} \approx 0. \tag{46}$$

The triplet $v^{\alpha\beta}$ satisfies the same Dirac brackets for both linear and non-linear cases:

$$\left\{v^{\alpha\beta}, v_{\lambda\mu}\right\}_{\text{DB}} = -\,i\,\delta^{\alpha}_{\lambda}\, v^{\beta}_{\mu}\,|v|. \tag{47}$$

One can check that they form the $su(2)$ algebra, where the square $v^2 = r^2$ is Casimir operator.

Coupling to chiral superfields

The non-linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ admits description through chiral superfields, then we couple it to the linear chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$. For what follows, it is convenient to deal with another form of the triplet V^{ij} given by

$$
\begin{aligned}
Y &= \frac{iR}{4}\left(R^2+V^2\right)\left[\frac{1}{(V^2-R^2+2iR\,V_{12})}-\frac{1}{(V^2-R^2-2iR\,V_{12})}\right],\\
U &= \frac{R^2\,V^{22}}{\sqrt{2}\,(V^2-R^2-2iR\,V_{12})}\,,\\
\bar{U} &= \frac{-\,R^2\,V^{11}}{\sqrt{2}\,(V^2-R^2+2iR\,V_{12})}\,.
\end{aligned}
\tag{48}
$$

The non-linear constraints become

$$
\begin{aligned}
\bar{D}_i U &= 0, \qquad D^i U = \frac{\sqrt{2}\,i}{R}\bar{D}^i\left[2Y^2-2U\bar{U}+i\,Y\left(R^2+8U\bar{U}-4Y^2\right)^{\frac{1}{2}}\right],\\
D^i\bar{U} &= 0, \qquad \bar{D}_i\bar{U} = \frac{\sqrt{2}\,i}{R}D_i\left[2Y^2-2U\bar{U}-i\,Y\left(R^2+8U\bar{U}-4Y^2\right)^{\frac{1}{2}}\right].
\end{aligned}
\tag{49}
$$

The chiral superfield U is solved by

$$\begin{aligned} U\left(t_{\mathrm{L}}\,,\theta_i\right) \;&=\; u+\sqrt{2}\,\theta_k\psi^k-\frac{1}{2\sqrt{2}\,R^2}\,\theta_k\theta^k\,C\left[R^2+8u\bar{u}-8y^2-4iy\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right] \\ &-\frac{i}{\sqrt{2}\,R}\,\theta_k\theta^k\,\dot{y}\left[\left(R^2+8u\bar{u}-4y^2\right)^{-\frac{1}{2}}\left(R^2+8u\bar{u}-8y^2\right)-4iy\right] \\ &-\frac{2\sqrt{2}\,\theta_k\theta^k\,\dot{u}\bar{u}}{R\left(R^2+8u\bar{u}\right)}\left[R^2+8u\bar{u}-8y^2-4iy\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right] \\ &+\frac{\sqrt{2}}{R}\,\theta_k\theta^k\left(u\dot{\bar{u}}+\dot{u}\bar{u}\right)\left[1-2iy\left(R^2+8u\bar{u}-4y^2\right)^{-\frac{1}{2}}\right]+\theta_k\theta^k\left(\psi^2\text{ term}\right). \end{aligned} \tag{50}$$

From plane to squashed sphere

In (E. Ivanov, S. S., *Phys. Rev. D* **105** (2022) 086027) we considered the simplest WZ Lagrangian for the linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ given by

$$\mathcal{L}_{\text{plane}} = \frac{C}{2}\,(c-y) + \frac{i}{2}\,(u\dot{\bar{u}} - \dot{u}\bar{u}) - \frac{1}{4}\,\chi^i_1\chi_{i2}\,, \tag{51}$$

where c is a real constant parameter. The constraint $y \approx c$ defines a non-commutative plane with Dirac bracket

$$\{u,\bar{u}\}_{\text{DB}} = i. \tag{52}$$

Here we show that a modification of the non-commutative plane for the non-linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ leads to a squashed 2-sphere. The relevant WZ action written as a superpotential is composed of two parts:

$$S_{\text{sq.sphere}} = \frac{iR}{8\sqrt{2}}\int dt_{\text{L}}\,d^2\theta\left(1+\frac{4ic}{R}\right)U - \frac{iR}{8\sqrt{2}}\int dt_{\text{R}}\,d^2\bar{\theta}\left(1-\frac{4ic}{R}\right)\bar{U}. \tag{53}$$

The action is invariant only under the $\text{U}(1)_{\text{rot.}}$ rotation from $\text{SO}(4)$.

The component Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\rm sq.sphere} \;=\;& \frac{iu\dot{\bar{u}}}{2\left(R^2+8u\bar{u}\right)}\left(1-\frac{4ic}{R}\right)\left[R^2+8u\bar{u}-8y^2+4iy\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]\\ &-\frac{i\dot{u}\bar{u}}{2\left(R^2+8u\bar{u}\right)}\left(1+\frac{4ic}{R}\right)\left[R^2+8u\bar{u}-8y^2-4iy\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]\\ &+\frac{C}{2R}\left[\frac{c}{R}\left(R^2+8u\bar{u}-8y^2\right)-y\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]+\psi^2\ \text{term}.\end{aligned} \tag{54}$$

One can see that it contains the FI part. The constraint imposed by the equation of motion for C reads

$$\frac{1}{2R}\left[\frac{c}{R}\left(R^2+8u\bar{u}-8y^2\right)-y\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]\approx 0. \tag{55}$$

Dirac brackets are

$$\begin{aligned}\{u,\bar{u}\} &= \frac{i}{R}\left(R^2+8u\bar{u}\right)\left(R^2+8u\bar{u}-4y^2\right)^{-\frac{1}{2}},\\ \{y,u\} &= \frac{4iu}{R}\left[y\left(R^2+8u\bar{u}-4y^2\right)^{-\frac{1}{2}}-\frac{2c}{R}\right]\left(R^2+8u\bar{u}\right)\left[R^2+8u\bar{u}-8y^2+\frac{16cy}{R}\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]^{-1},\\ \{\bar{u},y\} &= \frac{4i\bar{u}}{R}\left[y\left(R^2+8u\bar{u}-4y^2\right)^{-\frac{1}{2}}-\frac{2c}{R}\right]\left(R^2+8u\bar{u}\right)\left[R^2+8u\bar{u}-8y^2+\frac{16cy}{R}\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]^{-1}.\end{aligned}$$

The constraint is rewritten as

$$\frac{R^2\left(R+\sqrt{R^2+16c^2}\right)}{8\left[(x_3)^2+(x_4)^2\right]^2}\left[x_3-\frac{4cx_4}{R+\sqrt{R^2+16c^2}}\right]\left[x_4+\frac{4cx_3}{R+\sqrt{R^2+16c^2}}\right]\approx 0, \tag{56}$$

and has two equivalent solutions corresponding to planes:

$$1)\quad x_3\approx\frac{4cx_4}{R+\sqrt{R^2+16c^2}}\,,\qquad 2)\quad x_4\approx-\frac{4cx_3}{R+\sqrt{R^2+16c^2}}\,. \tag{57}$$

Without loss of generality, we can choose the first solution that implies a plane section of the sphere S^3. Hence the constraint defines a squashed 2-sphere:

$$\begin{aligned}1)\quad \left[R^2-(x_1)^2-(x_2)^2-(x_4)^2\right]^{\frac{1}{2}}\approx\frac{4|cx_4|}{R+\sqrt{R^2+16c^2}}\quad\Rightarrow\\ \Rightarrow\quad (x_1)^2+(x_2)^2+\frac{2\sqrt{R^2+16c^2}}{R+\sqrt{R^2+16c^2}}(x_4)^2\approx R^2.\end{aligned} \tag{58}$$

The constraint respects only $\mathrm{U}(1)$ symmetry from $\mathrm{SO}(4)$ that keeps invariant the bilinear form $(x_1)^2+(x_2)^2$.

We proceed to the coupling of the linear multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ to the non-linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$. The linear multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by the chiral superfield Z:

$$\bar{D}_i Z = 0, \qquad Z\left(t_{\rm L}, \theta_i\right) = z + \sqrt{2}\,\theta_k \xi^k + \theta_k \theta^k B. \tag{59}$$

Let us consider the simplest interaction $U\,Z$ together with the squashed sphere action and the kinetic term $K\left(Z, \bar{Z}\right)$:

$$\begin{aligned} S \;=\;& \frac{1}{4}\int dt\, d\theta^2\, d^2\bar{\theta}\, K\left(Z,\bar{Z}\right) + \frac{iR}{8\sqrt{2}}\int dt_{\rm L}\, d^2\theta \left(1 + \frac{4ic}{R}\right) U - \frac{iR}{8\sqrt{2}}\int dt_{\rm R}\, d^2\bar{\theta}\left(1 - \frac{4ic}{R}\right)\bar{U} \\ & + \frac{\mu}{2}\int dt_{\rm L}\, d^2\theta\, Z\, U + \frac{\mu}{2}\int dt_{\rm R}\, d^2\bar{\theta}\, \bar{Z}\,\bar{U}. \end{aligned} \tag{60}$$

We will limit our consideration to the bosonic Lagrangian which is given by (up to full time derivatives)

$$
\begin{aligned}
\mathcal{L} \;=\; & \left(\dot{\bar z}\dot z + \bar B B\right)\partial_z\partial_{\bar z}K\left(z,\bar z\right) + \mu\left(B\,u + \bar B\,\bar u\right) \\
& + \frac{C}{2R^2}\left\{\left[c - \frac{\mu}{\sqrt{2}}\left(z+\bar z\right)\right]\left(R^2 + 8u\bar u - 8y^2\right) - R\,y\left(R^2 + 8u\bar u - 4y^2\right)^{\frac{1}{2}}\left[1 - \frac{2\sqrt{2}\,i\mu}{R}\left(z-\bar z\right)\right]\right\} \\
& + \frac{iu\dot{\bar u}}{2\left(R^2+8u\bar u\right)}\left(1 - \frac{4ic}{R} + \frac{4\sqrt{2}\,i\mu}{R}\,\bar z\right)\left[R^2 + 8u\bar u - 8y^2 + 4iy\left(R^2 + 8u\bar u - 4y^2\right)^{\frac{1}{2}}\right] \\
& - \frac{i\dot u\bar u}{2\left(R^2+8u\bar u\right)}\left(1 + \frac{4ic}{R} - \frac{4\sqrt{2}\,i\mu}{R}\,z\right)\left[R^2 + 8u\bar u - 8y^2 - 4iy\left(R^2 + 8u\bar u - 4y^2\right)^{\frac{1}{2}}\right] \\
& - \frac{i}{32}\left(1 - \frac{4ic}{R} + \frac{4\sqrt{2}\,i\mu}{R}\,\bar z\right)\partial_t\left[R^2 + 8u\bar u - 8y^2 + 4iy\left(R^2 + 8u\bar u - 4y^2\right)^{\frac{1}{2}}\right] \\
& + \frac{i}{32}\left(1 + \frac{4ic}{R} - \frac{4\sqrt{2}\,i\mu}{R}\,z\right)\partial_t\left[R^2 + 8u\bar u - 8y^2 - 4iy\left(R^2 + 8u\bar u - 4y^2\right)^{\frac{1}{2}}\right]. \qquad (61)
\end{aligned}
$$

The equation of motion for C imposes that

$$\left[1-\frac{2\sqrt{2}\,i\mu}{R}\left(z-\bar{z}\right)\right]\left[\frac{\tilde{c}}{2R^2}\left(R^2+8u\bar{u}-8y^2\right)-\frac{y}{2R}\left(R^2+8u\bar{u}-4y^2\right)^{\frac{1}{2}}\right]\approx 0, \tag{62}$$

where

$$\tilde{c}\left(z,\bar{z}\right)=\left[c-\frac{\mu}{\sqrt{2}}\left(z+\bar{z}\right)\right]\left[1-\frac{2\sqrt{2}\,i\mu}{R}\left(z-\bar{z}\right)\right]^{-1}. \tag{63}$$

This is the same constraint of the squashed sphere S^2 with a parameter $\tilde{c}$ depending on the coordinates z and $\bar{z}$. Therefore, the squashed 2-sphere is defined by the parameter $\tilde{c}$ set at each point of Kähler manifold.

Summary

- We considered the non-linear version of the mirror multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ as a semi-dynamical multiplet and constructed its action.
- We showed that the quadratic constraint for the superfield function induces the Laplace-Beltrami equation on the sphere S^3.
- We considered embeddings of round and squashed 2-dimensional spheres into S^3.
- We coupled the squashed sphere S^2 to the dynamical mirror multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ and constructed their interaction as a superpotential.
- In the limit $R \to \infty$, models of the linear multiplet are reproduced.

Outlook

- It would be interesting to find a solution of the torus $S^1 \times S^1$ as an embedding in S^3.
- Unfortunately, it is not clear how to couple the non-linear multiplet to the dynamical multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$. Perhaps, it can be done only within the harmonic superspace description.
- The next tempting problem is to consider the non-linear multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ as a semi-dynamical one (F. Delduc, E. Ivanov, *Nucl. Phys. B* **753** (2006) 211-241; *Nucl. Phys. B* **855** (2012) 815-853).
- There is also interest in the dynamical non-linear multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$. For example, one can consider a model of 3D supersymmetric particle in the field of a monopole (S. Bellucci, S. Krivonos, A. Sutulin, *Phys. Rev. D* **81** (2010) 105026, E. Ivanov, M. Konyushikhin, *Phys. Rev. D* **82** (2010) 085014) with the modified potentials

$$\mathcal{U}(v) = -\frac{1}{2}\left(\frac{1}{|v|} - \frac{|v|}{R^2}\right), \qquad \mathcal{A}_{\alpha\beta} = -\frac{k^{\gamma}_{(\alpha} v_{\beta)\gamma}}{|v|\,(2|k||v| - k^{\lambda\mu} v_{\lambda\mu})} - \frac{iR\, v_{\alpha\beta}}{|v|\,(R^2 + v^2)}\,. \tag{64}$$

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Thank you for your attention!