Series of representations

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Introduction

- Given a representation V, we consider the centraliser algebras, $A(n) = \mathrm{End}(\otimes^n V)$.
- The seminal examples are the symmetric group algebras and the Brauer algebras.
- The idea of a series is that the algebras depend on one or more continuous parameters and there are special values where we truncate to get the centraliser algebras.
- The series in this talk are finite and the Cartan types vary.
- The algebras are only known for $n \leqslant 5$.

Assumption

Let $L(\lambda)$ have highest weight λ. The decomposition of $L(\lambda) \otimes L(\lambda)$ is

$$\bigwedge^2 L(\lambda) \cong L(\theta) \oplus L(\mu)$$

$$S^2 L(\lambda) \cong L(0) \oplus L(\nu_1) \oplus \cdots \oplus L(\nu_k)$$

where 0 is the zero weight so $L(0)$ is the trivial representation and θ is the highest root so $L(\theta)$ is the adjoint representation.

Magic square

The Freudenthal magic square is the following square of Lie algebras. Each entry has a preferred representation.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	B_1^3	A_2^8	C_3^{21}	F_4^{52}
$\mathbb{C}$	A_2^8	$2A_2^{16}$	A_5^{35}	E_6^{78}
$\mathbb{H}$	C_3^{21}	A_5^{35}	D_6^{66}	E_7^{133}
$\mathbb{O}$	F_4^{52}	E_6^{78}	E_7^{133}	E_8^{248}

- The third (or $\mathbb{H}$) row representations satisfy the assumption with $k = 1$.
- The first (or $\mathbb{R}$) row and the fourth (or $\mathbb{O}$) row representations satisfy the assumption with $k = 2$.

Idea, Pierre Vogel

Take the list of Casimir values

$$[C(\lambda), C(\theta), C(\mu), C(\nu_1), \ldots, C(\nu_k)]$$

- Consider these as projective coordinates of a point in projective space $P^{k+2}(\mathbb{Q})$.
- There are unexpected linear relations; these are points in projective space $P^k(\mathbb{Q})$.

Strategy, Hans Wenzl

- Interpolate Casimirs/eigenvalues
- Construct representation of braid group, B_3.
- Determine structure constants of algebra $A(2)$.
- (Optional) Take limit $q \to 1$.

Quaternion row

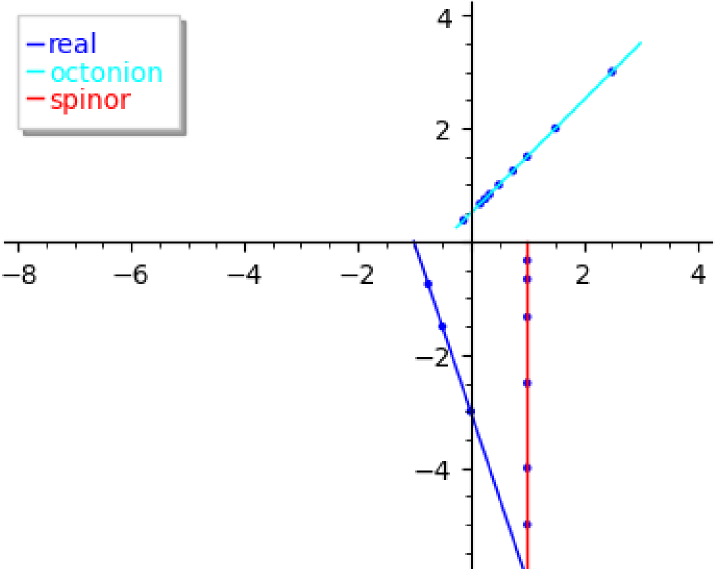
This gives Freudenthal triple sytems.

m	-2/3	0	1	2	4	8
$\mathfrak{g}$	A_1	$3A_1$	C_3	A_5	D_6	E_7
G	$\mathrm{SL}(2)$	$\mathrm{SL}(2) \times \mathfrak{S}_3$	$\mathrm{Sp}(6)$	$\mathfrak{S}_2 \times \mathrm{SL}(6)/\mu_2$	$\mathrm{Spin}(12)$	E_7
λ	$3\omega_1$	$\omega_1 + \omega_2 + \omega_3$	ω_3	ω_3	ω_6	ω_7

m	-3	-8/3	-5/2
$\mathfrak{g}$	D_5	B_3	G_2
G	$\mathrm{SO}(10)$	$\mathrm{Spin}(7)$	G_2
ω	ω_1	ω_3	ω_1

Plane

The case $k = 2$ gives a plane which contains three lines of representations.

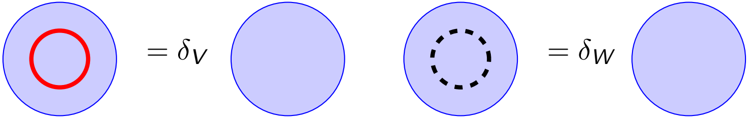


Symmetric spaces

Associated to a symmetric space is a Lie algebra with an involution. The $+1$-eigenspace is a Lie algebra and the -1-eigenspace is an L-module, V.

$2/3$	1	$4/3$	$8/3$	5	8	10	12
EVIII	EV	E1	A1		BD1	FII	EIV
E_8	E_7	E_6	A_2	$OSp(1\|2)$	D_4	F_4	E_6
D_8	A_7^*	C_4	A_1	A_1	B_3	B_4	F_4

Dimensions



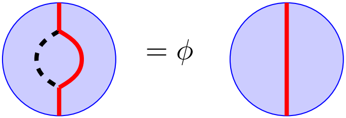
These dimensions have the values:

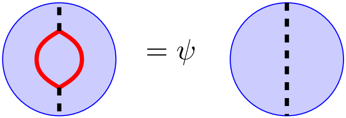
$$\delta_V = (-2)\frac{(2m+3n+3p)(3m+3n+2p)}{pm}$$

$$\delta_W = \frac{9(m+n+p)(2m+2n+p)(2m+3n+2p)(2m+3n+3p)}{mp^2(m-p)}$$

Bigons

The bigon coefficients are defined by





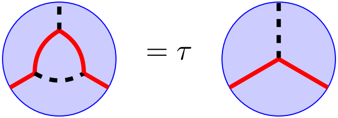
These are given the values

$$\phi = (-3)\frac{(m+n+p)(2m+3n+2p)}{mp}$$
$$\psi = \frac{2}{3}\frac{(m-p)(3m+3n+2p)}{m(2m+2n+p)}$$

Note that $\phi\delta_V = \psi\delta_W$.

Triangle

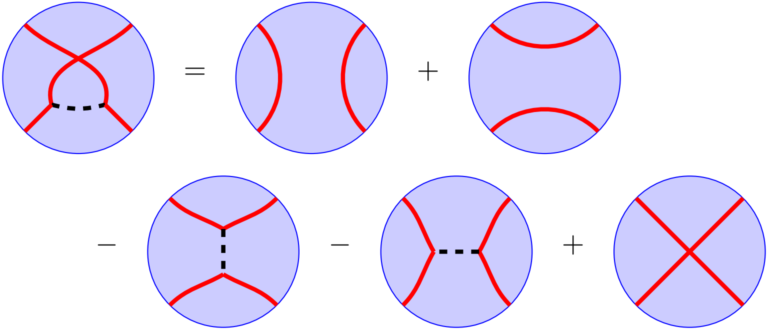
The triangle coefficient is defined by



This is given the value

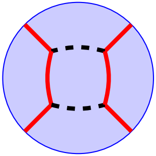
$$\tau = \frac{(3m^2 + 3mn + 4mp + 3np + 2p^2)}{3m(2m + 2n + p)}$$

HS relation

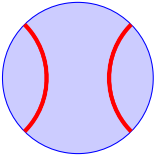


Square relation

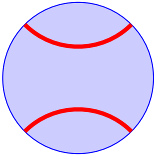
The coefficients $Sq_I, Sq_U, Sq_K, Sq_H, Sq_S$ are defined by



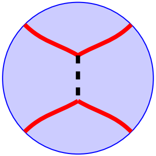
$= Sq_I$



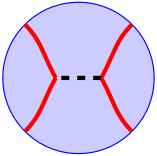
$+Sq_U$



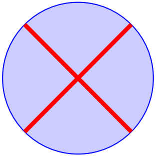
$+Sq_K$



$+Sq_H$



$+Sq_S$



Coefficients

These coefficients are given the values

$$Sq_I = \frac{(5m^2 + 8mn + 3n^2 + 4mp + 2np)}{2m(2m + 2n + p)}$$

$$Sq_U = (-1)\frac{(4m^2 + 10mn + 6n^2 + 5mp + 7np + 2p^2)(3m + 3n + 2p)}{2mp(2m + 2n + p)}$$

$$Sq_K = \frac{(4m + 3n + 2p)(3m + 3n + 2p)}{3m(2m + 2n + p)}$$

$$Sq_H = (-1)\frac{(5m^2 + 11mn + 6n^2 + 7mp + 7np + 2p^2)}{m(2m + 2n + p)}$$

$$Sq_S = \frac{(3m + 3n + 2p)^2}{2m(2m + 2n + p)}$$

Conclusion

- Most of the representations admit an R-matrix.
- The exceptions are the adjoint representations where $L(\theta) \oplus \mathbb{C}$ admits an R-matrix.
- For most lines the representations have the same tensor product graph.
- Do the lines correspond to families of integrable quantum field theories?