On the convex structure of the set of absolutely separable states and their relation to Wigner positivity

V. Abgaryan, A. Khvedelidze

August 20, 2023

Introduction

A brief anthology of quantum entanglement starting with Adam and Eve

- The superposition principle together with "a priori"-ty of the knowledge on the compositness of quantum systems enforces a tensor product structure on the quantum mechanical state space.
- This structure inevitably gives rise to states which can not be represented as tensor products of constituents' states. This in its right, predicts a plethora of purely quantum correlations between local degrees of freedom which are intractable from classical point of view (e.g. entanglement).
- From this point onward there is no reconciliation between quantum mechanics and classical perception of the physical world. However, there is a mighty spoon of quantum ~~tar~~ honey in the barrel of ~~honey~~ constructiveness tar.

Introduction

Honey

Within the framework of quantum information theory entanglement is considered as a valuable resource and its presence allows to perform non trivial protocols of quantum information communication and processing. Probably two most famous of this being

- Quantum teleportation: due to the "nonloclaity" of the phenomenon;
- Quantum computing: due to the quantum parallelism which allows to extract some aggregated information on the parallely performed computation.

Introduction: Formal definitions

Definition (Pure state separability/etnaglement)

Let the Hilbert space associated with a composite quantum system be given as $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\mathcal{H}_A$ and $\mathcal{H}_B$ being the states spaces associated with the components of the system. If a state $|\psi_{AB}\rangle \in \mathcal{H}_{AB}$ may be factorized as

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \text{, for some local } |\psi_{A,B}\rangle \in \mathcal{H}_{A,B}$$

then we call the pure state separable, otherwise it is said that the state is entangled.

Formal definitions

Definition (Mixed state separability/entanglement)

Let ρ_{AB}, be the density operator [a] *representing the mixed state of a composite quantum system, then it is called separable if there exists some local bases for the components such that it is possible to perform the following decomposition*

$$\rho_{AB} = \sum_i \omega_i \rho_A^i \otimes \rho_B^i \,, \textit{ with } \sum_i \omega_i = 1 \,, \omega_i \geq 0 \textit{ and } \rho_{A,B} \in \mathfrak{P}_{A,B} \,.$$

otherwise the mixed state is said to be entangled.

[a] The operator $\rho : \mathcal{H} \longrightarrow \mathcal{H}$ is called a density operator if $\rho_{\mathcal{H}} \in \mathfrak{P} = \left\{ \rho \in \mathcal{L}[\mathcal{H}] \,|\, \rho^\dagger = \rho \,, \mathrm{tr}[\rho] = 1 \,, \rho \geq 0 \right\}$.

Formal definitions

Remark

The separability (entanglement) property is invariant under local unitary transformations

$$\rho_{AB} \longrightarrow (U_A \otimes U_B)\ \rho_{AB}\,(U_A^\dagger \otimes U_B^\dagger)\,, \textit{with}\ U_{A,B} \in SU(\mathrm{dim}\mathcal{H}_{A,B})\,.$$

Remark

Immediately from the last definition it follows that the set of separable states

- i *is convex* [a] *,*
- ii *is the convex hull* [b] *of separable pure states,*
- iii *the later constitute the set of extreme*[c] *points of separable states.*

[a] The set $\mathcal{S} \subset \mathbb{V}$ is said to be convex if $\forall x, y \in \mathcal{S}$ and $t \in [0,1]$, $t\,x + (1-t)y \in \mathcal{S}$.

[b] The convex hull of a set $\mathcal{S}$ is defined as the set $\mathrm{convhull}(\mathcal{S}) = \{\sum_i \lambda_i s_i | \forall s_i \in \mathcal{S}\,, \lambda_i \geq 0\,, \sum_i \lambda_i = 1\}$.

[c] Let $\mathcal{S}$ be a convex set, a point $s \in \mathcal{S}$ is an extreme point for the later if $(\exists t \in (0,1)$ and $s_1, s_2 \in \mathcal{S}$, such that $s = t\,s_1 + (1-t)s_2) \implies (s = s_1 = s_2)$.

Posing the some questions

Question (a)

Are there such (absolutly) separable states which remain such under the action of global unitary transformations

$$\rho_{AB} \longrightarrow U_{AB}\, \rho\, U_{AB}^{\dagger}, \; where\,, U \in SU(\mathrm{dim}\mathcal{H}_{AB})\,?$$

Question (b)

What are the extreme points of this new set (these points will be regarded as "minimal" generating set in accordance with Krein-Milmans theorem, which loosely speaking states that a closed convex set is identical with the closeure of the convex hull of its extereme points)?

Question (c)

The aforementioned property of absolute separability is defined as a global unitary invariant property. What is it relation to other unitary invariant classicallity measures (specifically to Wigner Positivity)?

Answears?

The answer to the question (a)

is at least partially known. It is established, that for absolute separability APPT is necessary. For $2 \otimes n$ dimensional systems it is also known that this condition is sufficient and representable in a concise form

$$\lambda_1 - \lambda_{2n-1} \leq 2\sqrt{\lambda_{2n-2}\lambda_{2n}}$$

where $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{2n}$, are the eigenvalues of the density operator representing the given state of the composite system.

Questions (b) and (c) are the focus of this report.

Partial answer to question (b)

The current status of Question (b)

It has been established that for system of dimension $2 \otimes n$

- The absolutely separable states of $rank = 2n - 1$ are extreme points of the set. These are the points where the Hilbert-Schmidt separable ball touches the boundary of **AS.**
- By the way of counterexampling it has been demonstrated, that there are boundary points of **AS** which are not extreme.

That's about it!

In this report we hope to present an exhausting answer to the Question (b) for $2 \otimes n$ dimensional systems![1]

[1] It is possible to extend the proposed approach for the description of extreme points of arbitrary dimensional $APPT$ set, but it is debatable weather the game is worth the candle.

Getting started with the comprehensive answer to Question (b).

A direct consequence of Lidsky-Gelfand-Berezin theorem is the following

Lemma

Let ρ_{AB} be a density operator of an n-dimensional composite system, let it additionally be known that

$$\rho = t\rho_1 + (1-t)\rho_2\,, \quad \text{for some} \quad t \in [0,1]\,,$$

the there exist ϱ, ϱ_1 and ϱ_2 simultaneously diagonal operators with the same ordering of eigenvectors and eigenvalues, and identical with corresponding ρ's spectrum; and $c_i, \tilde{c}_i$ $(i = 1 \cdots n!)$ with $\sum_i c_i = \sum_i \tilde{c}_i = 1$ such that both of the following are true

$$\varrho = t\varrho_1 + (1-t)\sum_{P_i \in S_n} c_i P_i \varrho_2 P_i^\dagger\,,$$

$$\varrho = t\sum_{P_i \in S_n} \tilde{c}_i P_i \varrho_1 P_i^\dagger + (1-t)\varrho_2\,.$$

Reducing the problem of finding the extreme points to a problem in the unitary orbit space

Corollary

If $\mathcal{S} \subset \mathfrak{P}_n$ is convex, compact and is defined by some restriction on the global unitary invariants of the density operator, then $s \in \mathcal{S}$ is an extreme point of $\mathcal{S}$ iff its diagonal form is an extreme point of the set which is the intersection of the eigenvalue simplex Δ_n with the original restriction.

Remark

If the aforementioned restriction is not linear in eigenvalues, then $s \in \partial\mathcal{S} \cap \Delta_n$ is an extreme point of $\mathcal{S}$ iff it is an extreme point of $\mathcal{S} \cap \mathcal{O}[\mathfrak{P}_n]$, where the last is the unitary orbit space of $\mathfrak{P}_n$.

Remark

Hence, $s \in \mathfrak{P}_{AS}$ is an extreme point of $\mathcal{S}$ iff it is on the unitary orbit of extreem points of $\mathcal{O}[\mathfrak{P}_n]$.

Extreme points of absolutely separable states of a pair of qubits via $ext\,\{\partial\mathfrak{P}_{AS} \cap \mathcal{O}[\mathfrak{P}4]\}$

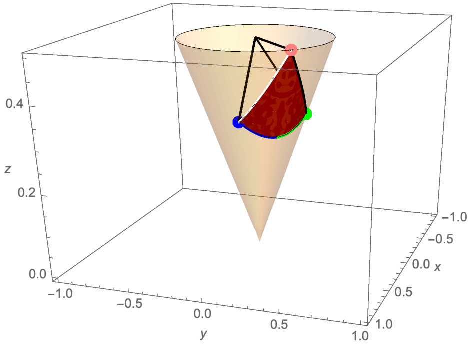

Figure: The cone of absolute separability intersecting the linear conus fixing a Weyl chamber.

Final answer

Proposition

$\rho \in \mathfrak{P}_{AS} \subset \mathfrak{P}_n$ is an extreme point of $\mathfrak{P}_{AS}$ if its isotropy group is

$$SU(k_1) \otimes SU(k_2) \otimes \cdots \otimes SU(k_m)\,,$$

with $2 \leq k_i \leq n-1$, for $i = 1 \cdots m$ and $\sum_i^m k_i \geq n-2$.

Question (b)

Proposition

Among the Stratonovich-Weyl kernels there are such that $\mathfrak{P}_{WP} \subset \mathfrak{P}_{AS}$.

Thank You