Hydrodynamics at Causal Boundaries, Examples in 3d Gravity

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Based on my recent work with
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arXiv:2305.01009

RDP School and Workshop
on
Mathematical Physics
Yerevan, Armenia

August 20, 2023

Outline

- **General relativity in the presence of boundaries**
- **Focus on causal (timelike and null) boundaries**
- **Work out boundary theories, symmetries and associated energy momentum tensor**
- **Hydrodynamics description**
- **Summary and outlook**

GR and Boundary symmetries

Presence of boundaries in spacetime brings in boundary d.o.fs

- **Any boundary:** Asymptotic boundary or any arbitrary codimensiton one surface in spacetime
- **Surface charges:** In diffeomorphic invariance boundary theories, non-trivial diffeomorphic transformations results in associated surface charges.
- **Boundary vs. Bulk:** We focus on the boundary instead of the usual viewpoint which focuses on the bulk.

Constructing the bulk they using the boundary theory, generalizing AdS/CFT, hopefully constructing a quantum theory at the boundary

General features of GR

- **A generally covariant theory**
- **Physical observables:** They are defined through local deffeomorphism invariant quantities,
- **Diffeomorphisms:** any two metric tensors related by diffeomorphisms are physically equivalent,

$$x^\mu \to x^\mu + \xi^\mu(x), \qquad g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu},\ \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla \nu \xi_\mu$$

GR and Boundary symmetries

General structure of EoM and d.o.fs

- **Metric:** In D dimensional spacetime, it has $D(D+1)/2$ components, $D(D-3)/2$ propagating modes, D deffeos,
- **Field equations:** D(D+1)/2 field equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, D(D-1)/2 are second order differential equations, D constraints
- **Bulk solutions:** Solutions can be fully specified by boundary and/or initial data, which in the most general case involves 2D functions over codimension one boundary,

Causal boundary

We take Gaussian-null-type coordinate system as the metric parametrization

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -Vdv^2 + \eta dvdr + \mathcal{R}^2(d\phi + Udv)^2 \tag{1}$$

V, U, $\mathcal{R}$ are functions of v, r, ϕ, while $\eta > 0$ is a function of v, ϕ

We take boundary $\mathcal{C}_r$ to be at constant r (with arbitrary r) surface and restrict ourselves to $V \geq 0$ surface

Causal boundary

A causal boundary at arbitrary $r = r_0$

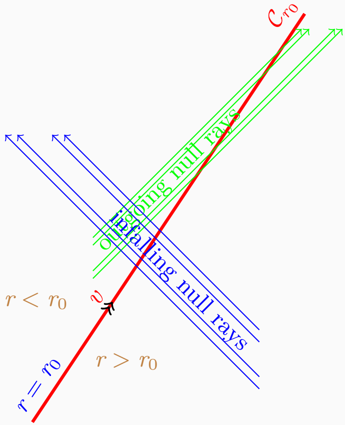


Figure 1: Depiction of a causal boundary at an arbitrary, we want to formulate physics in the outside $r \geq r_0$ region and excise $r \leq r_0$

Causal boundary

Boundary metric: The induced metric on $\mathcal{C}_r$ is then,

$$d\sigma^2 := \gamma_{ab}\, dx^a dx^b = -V dv^2 + \mathcal{R}^2 \left(d\phi + U dv\right)^2 , \qquad x^a = \{v, \phi\}. \quad (2)$$

Let s denote the vector field perpendicular to $\mathcal{C}_r$,

$$s_\mu dx^\mu := \frac{\eta}{\sqrt{V}} dr, \qquad s^\mu \partial_\mu = \frac{1}{\sqrt{V}} \left(\partial_v + \frac{V}{\eta}\partial_r - U\partial_\phi\right), \quad (3)$$

The induced metric $\gamma_{\mu\nu}$ can be written in terms of unit timelike vector field t^μ and spacelike vector field k^μ,

$$\gamma_{\mu\nu} = -t_\mu t_\nu + k_\mu k_\nu , \quad (4)$$

where

$$k_\mu dx^\mu := \mathcal{R}\left(d\phi + U dv\right) , \qquad k^\mu \partial_\mu = \frac{1}{\mathcal{R}}\, \partial_\phi , \quad (5)$$

$$t_\mu dx^\mu := -\sqrt{V}\left(dv - \frac{\eta}{V} dr\right) , \qquad t^\mu \partial_\mu = \frac{1}{\sqrt{V}} (\partial_v - U\partial_\phi) . \quad (6)$$

Causal boundary in terms of null vectors

The two spacelike and timelike vector fields s, t may be written in terms of linear combinations of two normalized null vector fields l, n,

$$t = \frac{1}{\sqrt{V}}\,l + \frac{1}{2}\,\sqrt{V}\,n\,, \qquad s = \frac{1}{\sqrt{V}}\,l - \frac{1}{2}\,\sqrt{V}\,n, \tag{7a}$$

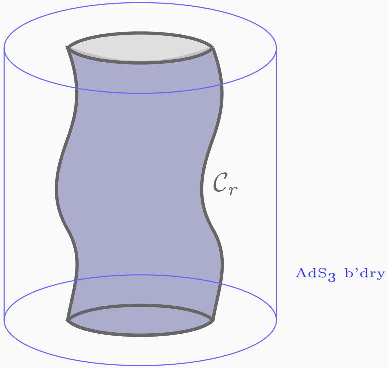
$$l = \frac{\sqrt{V}}{2}(t+s)\,, \qquad n = \frac{1}{\sqrt{V}}(t-s), \tag{7b}$$

with $l^2 = n^2 = 0$, and $l \cdot n = -1$, explicitly,

$$\begin{aligned} l_\mu x^\mu &= -\frac{1}{2}\,Vv + \eta\,, \qquad & n_\mu x^\mu &= -v\,, \\ l^\mu \partial_\mu &= \partial_v + \frac{V}{2\eta}\partial_r - U\partial_\phi\,, \qquad & n^\mu \partial_\mu &= -\frac{1}{\eta}\partial_r\,. \end{aligned} \tag{8}$$

Equation (7) also makes it clear that $\ln(\sqrt{V})$ may be viewed as a boost speed which acts on l, n like scaling by $\sqrt{V}, 1/\sqrt{V}$, respectively.

Causal boundary



Portion of AdS_3 bounded by a generic timelike boundary $\mathcal{C}_r$. We formulate physics in the shaded region. The wiggles on $\mathcal{C}_r$ are to highlight the boundary degrees of freedom, where the Brown-York-type charges $\mathcal{T}^{ab}$ are canonical conjugates of boundary metric components γ_{ab}.

Causal boundary

Geometrical invariantes of the boundary generating spacelike vector field s along the boundary:

$$
\begin{aligned}
\theta_s &:= q^{\alpha\beta}\nabla_\alpha s_\beta = \frac{1}{\sqrt{V}\mathcal{R}}\left(\mathrm{D}_v\mathcal{R} + \frac{V}{\eta}\partial_r\mathcal{R}\right)\,, \\
\omega_s &:= -k^\alpha t^\beta \nabla_\alpha s_\beta = -\frac{1}{2\mathcal{R}}\left(\frac{\mathcal{R}^2}{\eta}\partial_r U + \frac{\partial_\phi V}{V} - \frac{\partial_\phi \eta}{\eta}\right)\,, \\
\kappa_t &:= t^\beta t^\alpha \nabla_\alpha s_\beta = \frac{1}{2\sqrt{V}}\left(\frac{\mathrm{D}_v V}{V} - \frac{\partial_r V}{\eta} - 2\frac{\mathrm{D}_v \eta}{\eta}\right)\,.
\end{aligned}
\tag{9}
$$

By D_v we denote the derivatives along the v on $_r$,

$$
\mathrm{D}_v := \partial_v - \mathcal{L}_U, \tag{10}
$$

Similarly, for the two null vectors l, n the expansions θ_l, θ_n, the angular velocity, ω_l, and non-affinity parameter, κ, are given by

$$\begin{aligned}
\kappa &:= -l^\alpha n^\beta \nabla_\alpha l_\beta = \frac{\mathrm{D}_v \eta}{\eta} + \frac{\partial_r V}{2\eta}, \\
\omega_l &:= -k^\mu n^\nu \nabla_\mu l_\nu = -\frac{1}{2\mathcal{R}} \left(-\frac{\partial_\phi \eta}{\eta} + \frac{\mathcal{R}^2}{\eta} \partial_r U \right), \\
\theta_l &:= q_{\alpha\beta} \nabla^\alpha l^\beta = \frac{\mathrm{D}_v \mathcal{R}}{\mathcal{R}} + \frac{V}{2\eta} \frac{\partial_r \mathcal{R}}{\mathcal{R}}, \\
\theta_n &:= q_{\alpha\beta} \nabla^\alpha n^\beta = -\frac{1}{\eta} \frac{\partial_r \mathcal{R}}{\mathcal{R}}.
\end{aligned} \tag{11}$$

Field equations in 3D gravity

Field equations for Einstein-Λ theory are

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{12}$$

Straightforward computations show that one can solve for the r-dependence of the 3 functions in the metric (1) (recall that η is r-independent) obtained to be arXiv:2202.12129:

$$U = \mathcal{U} + \frac{1}{\lambda\mathcal{R}}\frac{\partial_\phi\eta}{\eta} + \frac{\Upsilon}{2\lambda\mathcal{R}^2}\,, \qquad \mathcal{R} = \Omega + r\,\eta\,\lambda\,, \tag{13a}$$

$$V = \frac{1}{\lambda^2}\left(-\Lambda\mathcal{R}^2 - \mathcal{M} + \frac{\Upsilon^2}{4\mathcal{R}^2} - \frac{2\mathcal{R}}{\eta}\mathcal{D}_v(\eta\lambda) + \frac{\Upsilon}{\mathcal{R}}\frac{\partial_\phi\eta}{\eta}\right)\,, \tag{13b}$$

where $\Omega, \lambda, \eta, \Upsilon, \mathcal{U}, \mathcal{M}$ are six functions of v, ϕ and $\mathcal{D}_v$ is defined in (10).

Field equations in 3D gravity

Einstein equations yield tow constraints/relations among the 6 codimension one functions of v, ϕ:

$$\mathrm{D}_v(\mathcal{R}^2\omega_s) + \mathcal{R}\partial_\phi(\sqrt{V}\kappa_t) + \mathcal{R}\theta_s\partial_\phi\sqrt{V} = 0\,, \tag{14a}$$

$$\mathrm{D}_v(\mathcal{R}\theta_s) + \kappa_t\mathrm{D}_v\mathcal{R} + \frac{1}{\sqrt{V}}\partial_\phi(V\omega_s) = 0\,. \tag{14b}$$

- The other equations $\mathcal{E}_{ss} = 0, \mathcal{E}_{ab} = 0$ are readily satisfied once (13) and (14) hold
- Equations (14) consist of two first-order time (v) derivative equations, which are linear in the variables θ_s, ω_s, and κ_t
- They are completely defined at the boundary $\mathcal{C}_r$
- The solution space is completely specified by 4 functions over $\mathcal{C}_r$

Causal boundary stress tensor

We start with extrinsic curvature of constant r surfaces $K_{\mu\nu}$,

$$K_{\mu\nu} := \frac{1}{2}\gamma^{\alpha}_{\mu}\gamma^{\beta}_{\nu}\mathcal{L}_s\gamma_{\alpha\beta} = \nabla_{(\mu}s_{\nu)} - s_{(\mu}s\cdot\nabla s_{\nu)}\,, \tag{15}$$

where $\gamma^{\alpha}_{\mu} = g^{\alpha\nu}\gamma_{\mu\nu}$.
and construct causal boundary Brown-York energy-momentum tensor as follwos

$$\mathcal{T}^{\mu\nu} = \frac{1}{8\pi G}\left(K^{\mu\nu} - K\gamma^{\mu\nu} + \frac{1}{\ell}\gamma^{\mu\nu}\right)\,, \qquad \ell^2 = -1/\Lambda, \tag{16}$$

- is by construction a symmetric tensor, $\mathcal{T}^{\mu\nu}s_{\nu} = 0$
- $\mathcal{T}^{\mu\nu}$ is defined on $\mathcal{C}_r$

Causal boundary stress tensor

It can hence be decomposed as

$$\mathcal{T}^{\mu\nu} = -\,\mathcal{E}\,(t^\mu t^\nu + k^\mu k^\nu) + 2\mathcal{J}\;k^{(\mu}t^{\nu)} + \frac{1}{2}\mathcal{T}\,(-t^\mu t^\nu + k^\mu k^\nu)\,, \qquad (17)$$

where we defined

$$\mathcal{E} := -\frac{1}{16\pi G}\;(\theta_s + \kappa_t)\;, \qquad \mathcal{T} := \frac{1}{8\pi G}\left(\kappa_t - \theta_s + \frac{2}{\ell}\right)\,, \qquad \mathcal{J} := \frac{\omega_s}{8\pi G}\,, \qquad (18)$$

where $\mathcal{T}$ is the trace of the causal boundary Brown-York energy-momentum tensor and $\theta_s, \omega_s, \kappa_t$ are defined in (9).
$\mathcal{T}^{\mu\nu}$ has only 3 non-zero components along the constant r surface and will be denoted by $\mathcal{T}^{ab} = \gamma^a_\mu \gamma^b_\nu \mathcal{T}^{\mu\nu}$.

Hydrodynamics description at finite distance: AdS_3 case

Field equations take the inspiring form,

$$\mathscr{D}_b \mathcal{T}^{ab} = 0\,, \tag{19}$$

where $\mathscr{D}_a$ is metric connection compatible with boundary metric γ_{ab}.
It suggests a hydrodynamics description with the following dictionary:
1) t^μ plays the role of fluid velocity field,
2)$\mathcal{E}$ corresponds to the fluid energy density,
3) $\mathcal{J}\, k_\mu$ related to the heat current (momentum flow), and
4) $\mathcal{T}\, k_\mu k_\nu$ is the corresponding dissipative tensor which is transverse to the fluid velocity direction.

Hydrodynamics description: Features

- These equations relate 2 out of 6 functions and hence the solution phase space is governed by 4 functions over $\mathcal{C}_r$,
- These equations relate 2 out of 6 functions and hence, as expected and discussed, the solution phase space is governed by 4 functions over $\mathcal{C}_r$.
- For any r at the boundary
- At $r \to \infty$ limit, where the boundary approaches the causal boundary of spacetime, we recover a conformal hydrodynamic description which only involves $2 + 2$ codimension 1 modes,
- All 3+3 modes in the configuration space appear in our hydrodynamic description on generic $\mathcal{C}_r$,

Some remarks

- Symplectic form then is

$$\Omega_{\mathcal{C}} = \int_{\mathcal{C}_r} d^2x \left[-\frac{1}{2}\delta(\sqrt{-\gamma}\,\mathcal{T}^{ab}) \wedge \delta\gamma_{ab} + \partial_a \delta Y_\circ^{ra}[g;\delta g] \right]. \qquad (20)$$

  in the absence of $Y_\circ$, the *off-shell* symplectic form consists of three causal boundary Brown-York charges $\mathcal{T}^{ab}$ which are canonically conjugate to the boundary metric γ_{ab}

This $3+3$ $(\gamma_{ab}, \mathcal{T}^{ab})$ decomposition of off-shell configuration space is different than $2+2$ $(\lambda^{-1}, \hat{\mathcal{M}}; \mathcal{U}, \hat{\Upsilon})$ plus $1+1$ (Ω, Π) decomposition

Conformal invariance of hydrodynamic desceription

Weyl scaling on ${}_r$, $\gamma_{ab} \to \mathcal{W}^2\,\gamma_{ab}$, is not a part of our boundary symmetries at generic r and hence the effective relativistic hydrodynamic description at the boundary is not a conformal one

- In general $\mathcal{T}^{ab}$ is divergence-free, but it is not traceless,
- In $r \to \infty$ limit, where the boundary approaches the causal boundary of spacetime,

We recover a conformal hydrodynamic description which

Other hydrodynamics frames

Consider two boundary metrics related by a Weyl scaling:

$$\gamma_{ab} \to \tilde{\gamma}_{ab} = \mathcal{W}^{-2}\,\gamma_{ab}\,, \tag{21}$$

where $\mathcal{W}$ is a generic function on the spacetime and a scalar in the $\tilde{\gamma}$–frame.
One can verify that,

$$\begin{aligned} \sqrt{-\gamma}\,\mathcal{T}^{ab}\;\delta\gamma_{ab} &= \sqrt{-\tilde{\gamma}}\,\tilde{\mathcal{T}}^{ab}\;\delta\tilde{\gamma}_{ab} + 2\,\sqrt{-\tilde{\gamma}}\,\tilde{\mathcal{T}}\;\frac{\delta\mathcal{W}}{\mathcal{W}} \\ \delta(\sqrt{-\gamma}\,\mathcal{T}^{ab}) \wedge \delta\gamma_{ab} &= \delta(\sqrt{-\tilde{\gamma}}\,\tilde{\mathcal{T}}^{ab}) \wedge \delta\tilde{\gamma}_{ab} + 2\,\delta(\sqrt{-\tilde{\gamma}}\;\tilde{\mathcal{T}}) \wedge \frac{\delta\mathcal{W}}{\mathcal{W}} \end{aligned} \tag{22}$$

where

$$\tilde{\mathcal{T}}^{ab} = \mathcal{W}^4\mathcal{T}^{ab}, \qquad \tilde{\mathcal{T}} := \tilde{\gamma}_{ab}\tilde{\mathcal{T}}^{ab} = \mathcal{W}^2\gamma_{ab}\mathcal{T}^{ab} := \mathcal{W}^2\mathcal{T}. \tag{23}$$

We raise and lower indices for tilde-quantities by $\tilde{\gamma}^{ab}$ and $\tilde{\gamma}_{ab}$ respectively, as such $\mathcal{T}_{ab} = \tilde{\mathcal{T}}_{ab}$

Divergence-free frames

The divergence-free condition (19) can be written as:

$$\tilde{\nabla}_b \tilde{\mathcal{T}}^{ab} = \frac{1}{2}\,\mathcal{T}\,\tilde{\nabla}^a \mathcal{W}^2 \tag{24}$$

where $\tilde{\nabla}_a$ is the covariant derivative w.r.t. $\tilde{\gamma}_{ab}$.
That is, in a generic Weyl-frame neither the divergence nor the trace of the energy-momentum tensor is zero.

Divergence-free frames

The above is true for an arbitrary Weyl factor $\mathcal{W}$. One may choose $\mathcal{W} = f(\mathcal{T})$, where f is an arbitrary function of $\mathcal{T}$. Then, one can construct a new divergence-free energy-momentum tensor T^{ab}

$$\tilde{\nabla}_a \mathrm{T}^{ab} = 0, \tag{25}$$

$$\mathrm{T}^{ab} := \tilde{\mathcal{T}}^{ab} - \frac{1}{2}\tilde{\gamma}^{ab} F(\mathcal{T}) F' = 2\mathcal{T} f f', \qquad \mathrm{T} = \tilde{\gamma}_{ab}\mathrm{T}^{ab} = \int^{\mathcal{T}} f^2 \mathcal{T}, \tag{26}$$

where *prime* denotes derivative w.r.t. the argument.

One may also show,

$$\delta(\sqrt{-\gamma}\,\mathcal{T}^{ab}) \wedge \delta\gamma_{ab} = \delta(\sqrt{-\tilde{\gamma}}\,\mathrm{T}^{ab}) \wedge \delta\tilde{\gamma}_{ab}\,. \tag{27a}$$

Weyl scalling by $\mathcal{W} = f(\mathcal{T})$ is a canonical transformation both off-shell and on-shell.

That is, the hydrodynamic description is not unique and since $f(\mathcal{T})$ is an arbitrary function, there are infinitely many such descriptions

Asymptotic boundary hydrodynamics

hydrodynamical description on $_r$ at a generic r, becomes more interesting when we take $r \to \infty$ and take $_\infty$ to be the usual AdS_3 causal (asymptotic) boundary

- This leads to a conformally invariant hydrodynamical description,
- At the asymptotic causal boundary we have an emergent conformal symmetry,
- In the hydrodynamic description,due to anomaly in either of Diff or Weyl parts of the symmetry algebra, the boundary stress tensor can be made either divergence-free or traceless, not both simultaneously.
- The anomalies associated with Weyl and diffeomorphisms at the asymptotic $2d$ cylinder can be transformed between these two slicings,

Null boundary hydrodynamics

Requiring that $_r$ at finite r is a null surface that amounts to having $V = 0$ at the position of the boundary. Requiring the null boundary $\mathcal{N}$ to be located at $r = 0$ yields

$$V(r=0) = 0 \qquad \Rightarrow \qquad \mathcal{M} = -\Lambda\Omega^2 + \frac{\Upsilon^2}{4\Omega^2} - \frac{2\,\Omega}{\eta}\mathcal{D}_v(\eta\lambda) + \frac{\Upsilon}{\Omega}\,\frac{\partial_\phi \eta}{\eta}\,. \quad (28)$$

we arrive at following equations which yields the desired null field equations

$$\bar{D}_v(\Omega^2 \bar{\omega}_l) - \Omega \partial_\phi \bar{\kappa} = 0\,, \quad (29a)$$

$$\bar{D}_v \bar{\theta}_l + (\bar{\theta}_l - \bar{\kappa})\bar{\theta}_l = 0\,, \quad (29b)$$

where $\bar{\kappa}, \bar{\omega}_l, \bar{\theta}_l$ are obtained from $\kappa, \omega_l, \theta_l$ at $r = 0$.

Null boundary hydrodynamics

To construct the hydrodynamic description at null boundaries we start from the definition of the shape operator or Weingarten map, 2109.11567, as

$$\mathbb{T}^a{}_b := -\frac{1}{8\pi G}\left(\mathbb{W}^a{}_b - \mathbb{W}\,\delta^a_b\right) . \tag{30}$$

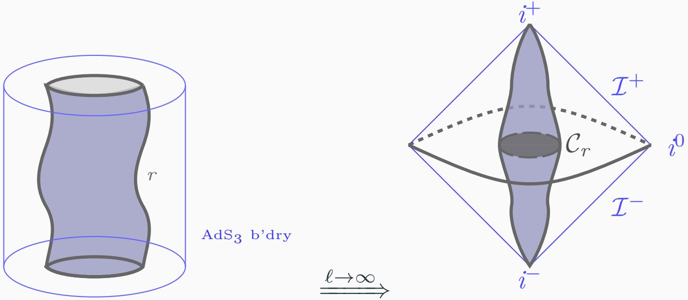
If the null boundary is spanned by null vector l^a and the spatial vector k^a, the Carrollian energy-momentum tensor is given by

$$\mathbb{T}^a{}_b = \frac{1}{8\pi G}\left[\bar{\kappa}\,\bar{k}^a\bar{k}_b - \bar{\omega}_l\,\bar{l}^a\bar{k}_b - \bar{\theta}_l\,\bar{l}^a\bar{n}_b\right] , \qquad \mathbb{T} := \mathbb{T}^a{}_a = \frac{1}{8\pi G}(\bar{\theta}_l + \bar{\kappa})\,. \tag{31}$$

where $\bar{\theta}_l$ is the expansion of the null surface, $\bar{\kappa}$ is its non-affinity parameter and $\bar{\omega}_l$ is its angular velocity.

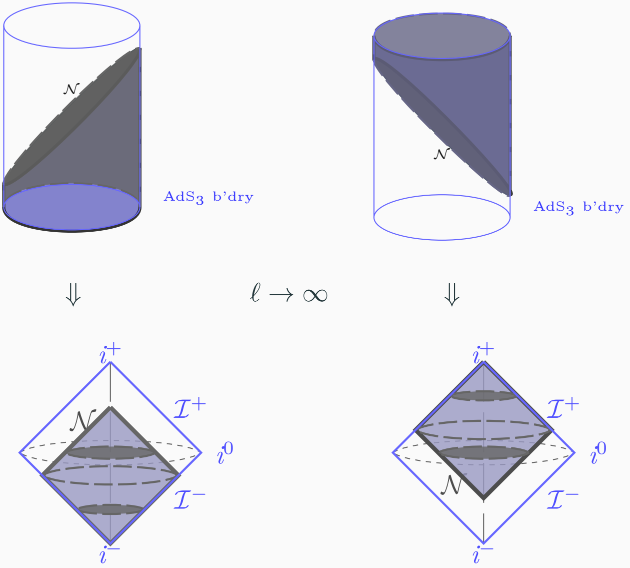
$$\mathbb{D}_a\mathbb{T}^a{}_b = \mathrm{P}^\nu{}_b\mathrm{P}^\alpha{}_\mu\nabla_\alpha\mathbb{T}^\mu{}_\nu\,. \tag{32}$$

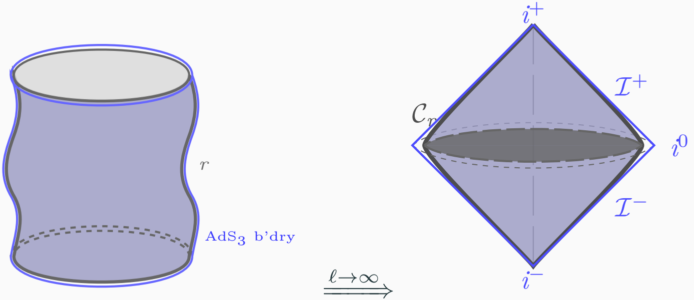
The boundary theory is a Carrollian theory,



Hydrodynamic description for flat case will obtain when $\Lambda \to 0$ limit of what we had in the AdS_3 case

Flat limit in null case





- Since we have, r-dependence, we may take $r \to \infty$ limit and obtain asymptotic AdS_3 hydrodynamics description, then take $\ell \to \infty$ limit, to obtain an asymptotic Carrollian hydrodynamics description.

Takeaway lessons so far

Boundaries bring in “boundary degrees of freedom”

- Solution space was obtained for r-dependence and the corresponding symplectic form for a time-like boundary in AdS_3 gravity,
- Boundary d.o.f may be labeled by surface charges associated with nontrivial diffeos,
- They accept a hydrodynamicd description at finite r and asymptotic,
- There is a regular limit for hydrodynamics at flat case,
- The description has been developed for null boundaries in $3d$,

Outlook

- Extension to higher dimensions. Study the role of bulk propagating modes,
- Probably a different hydrodynamics description (more analogue fluid constitutes), in progress
- Going deeper into the fluid/gravity correspondence, extending the paradigm for more general spacetimes,

Our main motivation

Understanding the boundary theory for gravity
and their effective descriptions

may help us to understand the nature of gravity
and its quantization

Thank you note

- Local organizers, Armen and Erik,
- IPM, Prof. Sheikh Jabbari (Supervisor), Prof. Farzan (Head of the physics department)
- ICTP,

Questions and Comments?

Untitled, 1968, Mark Rothko

Backup slides

Covariant phase space formulation

Consider a covariant Lagrangian together with a boundary term

$$L[\varphi] = L_0[\varphi] + \partial_\mu L^\mu_{\text{bdy}}[\varphi], \tag{33}$$

where φ denotes generic fields we have in the problem.
One can read symplectic potential,

$$\Theta^\mu[\varphi;\varphi] = \Theta^\mu_{\text{LW}}[\varphi;\delta\varphi] + \delta L^\mu_{\text{bdy}}[\varphi] + \partial_\nu Y^{\mu\nu}[\varphi;\delta\varphi], \tag{34}$$

where $\Theta^\mu_{\text{LW}}[\varphi;\delta\varphi]$ is the Lee-Wald symplectic potential [**?**] and $Y^{\mu\nu}$ is a skew-symmetric tensor density of weight +1

$$\delta L_0 \approx \partial_\mu \Theta^\mu_{\text{LW}}[\varphi;\delta\varphi] \tag{35}$$

The symplectic form is

Using the symplectic potential one can define the symplectic form (see [?] and appendix B of [?])

$$\Omega[\delta_1\varphi, \delta_2\varphi; \varphi] := \int_r^{D-1} x_\mu\, \omega^\mu[\delta_1\varphi, \delta_2\varphi; \varphi], \qquad \omega^\mu[\delta_1\varphi, \delta_2\varphi; \varphi] := \delta_1\Theta^\mu[\delta_2\varphi, \tag{36}$$

Given the symplectic potential one can compute the Hamiltonian generators (charge variations) associated with the symmetry generators ξ [?]:

$$\not\delta Q_\xi[\delta\varphi, \delta_\xi\varphi; \varphi] := \int_r^{D-1} x_\mu\, \omega^\mu[\delta\varphi, \delta_\xi\varphi; \varphi]. \tag{37}$$

By the fact that the symplectic current is conserved on-shell, $\partial_\mu\omega^\mu \approx 0$, and by virtue of the Poincaré lemma, $\omega^\mu[\delta\varphi, \delta_\xi\varphi; \varphi] = \partial_\nu\, \mathcal{Q}_\xi^{\mu\nu}[\delta\varphi; \varphi]$, we get

$$\not\delta Q_\xi = \oint_{\mathcal{C}_{r,v}} \mathcal{Q}_\xi^{\mu\nu}[\delta\varphi; \varphi] x_{\mu\nu} \tag{38}$$

Symplectic form

where

$$L^{\mu}_{\text{GHY}} := \frac{\sqrt{-g}}{8\pi G}\left(K - \frac{1}{\ell}\right)s^{\mu}\,, \qquad Y^{\mu\nu}_{\text{GHY}} := -\frac{\sqrt{-g}}{8\pi G}\, s^{[\mu}\delta s^{\nu]}\,. \tag{39}$$

With these choices the symplectic potential becomes

$$_{\mathcal{C}} = \int_{\mathcal{C}_r} {}^2x \left[-\frac{1}{2}\sqrt{-\gamma}\,\mathcal{T}^{ab}\delta\gamma_{ab} + \delta\left(s\cdot L_{\circ}\, s^{r}\right) + \partial_a Y^{ra}_{\circ}[g;\delta g]\right]\,. \tag{40}$$

Symplectic form then is

$$\Omega_{\mathcal{C}} = \int_{\mathcal{C}_r} {}^2x \left[-\frac{1}{2}\delta(\sqrt{-\gamma}\,\mathcal{T}^{ab}) \wedge \delta\gamma_{ab} + \partial_a \delta\, Y^{ra}_{\circ}[g;\delta g]\right]\,. \tag{41}$$

in the absence of $Y_{\circ}$, the *off-shell* symplectic form consists of three causal boundary Brown-York charges $\mathcal{T}^{ab}$ which are canonically conjugate to the boundary metric γ_{ab}

Algebra of charges

the on-shell solution space can be spanned by four codimension one charges $\hat{\mathcal{M}}, \hat{\Upsilon}, \Omega, \Pi$.
These charges satisfy $Vir \oplus Vir \oplus Heisengerg$ or $BMS_3 \oplus Heisenberg$ algebra, at Brown-Henneaux central charge:

$$\{\Omega(v,\phi), \Pi(v,\phi')\} = 16\pi G\, \delta(\phi - \phi') \tag{42a}$$

$$\{\hat{\Upsilon}(v,\phi), \hat{\Upsilon}(v,\phi')\} = 16\pi G \left(\hat{\Upsilon}(v,\phi')\partial_\phi - \hat{\Upsilon}(v,\phi)\partial_{\phi'}\right) \delta(\phi - \phi') \tag{42b}$$

$$\{\hat{\mathcal{M}}(v,\phi), \hat{\mathcal{M}}(v,\phi')\} = -16\pi G\Lambda \left(\hat{\Upsilon}(v,\phi')\partial_\phi - \hat{\Upsilon}(v,\phi)\partial_{\phi'}\right) \delta(\phi - \phi') \tag{42c}$$

$$\{\hat{\Upsilon}(v,\phi), \hat{\mathcal{M}}(v,\phi')\} = 16\pi G \left(\hat{\mathcal{M}}(v,\phi')\partial_\phi - \hat{\mathcal{M}}(v,\phi)\partial_{\phi'} - 2\partial_\phi^3\right) \delta(\phi - \phi') \tag{42d}$$