Propagating-wave approximation in potential scattering

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RDP School and Workshop on Mathematical Physics, 20 August 2023

Joint work with Ali Mostafazadeh (Koc University, Istanbul, Türkiye).

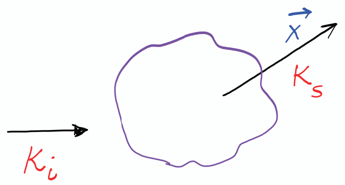
1. F. L. and A. Mostafazadeh, "Propagating-wave approximation in two-dimensional potential scattering," Phys. Rev. A **106** (2022) no.3, 032207 [arXiv:2204.05153 [quant-ph]].
2. F. L. and A. Mostafazadeh, "Existence of the transfer matrix for a class of nonlocal potentials in two dimensions," J. Phys. A **55** (2022) no.43, 435202 [arXiv:2207.10054 [math-ph]].

Overview

1. The Lippmann-Schwinger equation and the Born approximation.
2. The Propagating-wave approximation: a nonperturbative approximation scheme for performing scattering calculations in two and three dimensions.
3. Complex potentials for which this approximation scheme is exact.

Scattering in $D > 1$

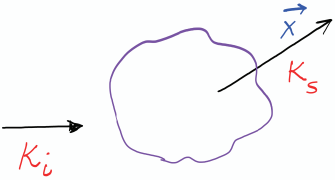
A linear scattering process in a $d \geq 2$ dimensional setup can be modeled by an incident plane waves that scatters off a short range potential $v : \mathbb{R}^d \to \mathbb{C}$.



The scattering data is encapsulated in the scattering amplitude f according to

$$\psi(\mathbf{x}) \to \langle \mathbf{x}|\mathbf{k}\rangle + (2\pi)^{-\frac{d}{2}} f(\mathbf{k}_s, \mathbf{k}) \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|^{\frac{d}{2}}} \quad \text{for} \quad \|\mathbf{x}\| \to \infty,$$

where $\mathbf{x}$ denotes the location of the detector, $\mathbf{k}_s := k\frac{\mathbf{x}}{\|\mathbf{x}\|}$, $k := \|\mathbf{k}\|$, and $\langle \mathbf{x}|\mathbf{k}\rangle = (2\pi)^{-d/2} e^{i\mathbf{k}\cdot\mathbf{x}}$.



The Lippmann-Schwinger equation

The scattered wave $\psi : \mathbb{R}^d \to \mathbb{C}$ solves the Lippmann-Schwinger equation

$$|\psi\rangle = |\mathbf{k}\rangle + G(\hat{\mathbf{p}})v(\hat{\mathbf{x}})\,|\psi\rangle\,,$$

where $G(\hat{\mathbf{p}}) := (k^2 - \hat{\mathbf{p}}^2 + i\varepsilon)^{-1}$, $\varepsilon \to^+ 0$ and

$$|\psi\rangle := \int_{\mathbb{R}^d} d\mathbf{x}\,|\mathbf{x}\rangle\,\psi(\mathbf{x}).$$

The Lippmann-Schwinger equation in the momentum representation reads

$$\langle \mathbf{p}|\psi\rangle = \langle \mathbf{p}|\mathbf{k}\rangle + G(\mathbf{p}) \int_{\mathbb{R}^d} d\mathbf{q}\, w(\mathbf{p}-\mathbf{q}) \langle \mathbf{q}|\psi\rangle \,,$$

where

$$w(\mathbf{p}-\mathbf{q}) := \langle \mathbf{p}|v(\hat{\mathbf{x}})|\mathbf{q}\rangle = (2\pi)^{-d}\tilde{v}(\mathbf{p}-\mathbf{q}),$$

and $\tilde{v} : \mathbb{R}^d \to \mathbb{C}$ denotes the Fourier transform of v.

Born series

This equation is solved iteratively by the Born series

$$\langle \mathbf{p} | \psi \rangle = \sum_{n=0} \left\langle \mathbf{p} | \psi^{(n)} \right\rangle ,$$

in which the zeroth term $\left\langle \mathbf{p} | \psi^{(0)} \right\rangle = \langle \mathbf{p} | \mathbf{k} \rangle$ and the n-th term is given by iteration

$$\left\langle \mathbf{p} | \psi^{(n)} \right\rangle = G(\mathbf{p}) \int_{\mathbb{R}^d} d\mathbf{q}\, w(\mathbf{p} - \mathbf{q}) \left\langle \mathbf{q} | \psi^{(n-1)} \right\rangle \quad \text{for} \quad n \in \mathbb{N}.$$

Asymptotically, i.e., for $\|\mathbf{x}\| \to \infty$,

$$\text{Schrödinger equation} \quad \to \quad \widehat{\mathbf{p}}^2 |\psi\rangle = k^2 |\psi\rangle$$

$$\psi(\mathbf{x}) \to \int_{\mathcal{S}_k} d\mathbf{p} \, \langle \mathbf{x}|\mathbf{p}\rangle \, \langle \mathbf{p}|\psi\rangle$$

To compute the scattering amplitude f we only need to compute $\langle \mathbf{p}|\psi\rangle$ for $\mathbf{p} \in \mathcal{S}_k := \{\mathbf{p}|\mathbf{p} \in \mathbb{R}^d, \|\mathbf{p}\| = k\}$.

Using

$$\int_{\mathcal{S}_k} d\mathbf{p} \langle \mathbf{x}|\mathbf{p}\rangle G(\mathbf{p})w(\mathbf{p}-\mathbf{k}) = \int d\mathbf{x}' \langle \mathbf{x}|G(\hat{\mathbf{p}})|\mathbf{x}'\rangle \int_{\mathcal{S}_k} d\mathbf{p} \langle \mathbf{x}'|\mathbf{p}\rangle w(\mathbf{p}-\mathbf{k}),$$

and

$$\langle \mathbf{x}|G(\hat{\mathbf{p}})|\mathbf{x}'\rangle \to (2\pi)^{\frac{d}{2}} N_d \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|^{\frac{d-1}{2}}} \langle \mathbf{k}_s|\mathbf{x}'\rangle \text{ for } \|\mathbf{x}\| \to \infty,$$

where in $d = 2$ and $d = 3$ dimensions N_d equals $N_2 = -\sqrt{\frac{i}{8\pi k}}$ and $N_3 = -\frac{1}{4\pi}$ respectively, we obtain

$$\psi^{(n)}(\mathbf{x}) \to (2\pi)^{\frac{d}{2}} N_d \frac{e^{ik\|\mathbf{x}\|}}{\|\mathbf{x}\|^{\frac{d}{2}}} \int_{\mathbb{R}^d} d\mathbf{q} w(\mathbf{k}_s - \mathbf{q}) \left\langle \mathbf{q}|\psi^{(n-1)}\right\rangle \text{ for } \|\mathbf{x}\| \to \infty.$$

Thus, the scattering amplitude is given by

$$f(\mathbf{k}_s, \mathbf{k}) = \sum_{n=1} f^{(n)}(\mathbf{k}_s, \mathbf{k}),$$

where

$$f^{(n)}(\mathbf{k}_s, \mathbf{k}) := N_d \int_{\mathbb{R}^d} d\mathbf{q}\, \tilde{v}(\mathbf{k}_s - \mathbf{q}) \left\langle \mathbf{q} | \psi^{(n-1)} \right\rangle .$$

The first Born approximation

In the first Born approximation

$$\langle \mathbf{p} | \psi \rangle = \langle \mathbf{p} | \mathbf{k} \rangle + G(\mathbf{p}) w(\mathbf{p} - \mathbf{k}),$$

we have

$$f(\mathbf{k}_s, \mathbf{k}) \approx f^{(1)}(\mathbf{k}_s, \mathbf{k}) = N_d \, \tilde{v}(\mathbf{k}_s - \mathbf{k}).$$

Since

$$\mathbf{k}_s - \mathbf{k} \in \mathscr{D}_{2k} = \{\mathbf{q} | \, \|\mathbf{q}\| \leq 2k\}$$

we conclude that

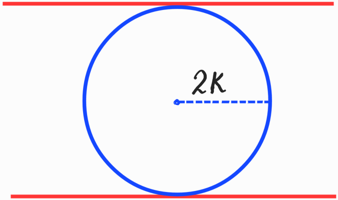
1. Fourier component of the potential $\tilde{v}(\mathbf{p})$ does not contribute in f if $\|\mathbf{p}\| > 2k$.
2. The scattering amplitudes of the potentials v_1 and v_2 are indistinguishable at wave-numbers $k' < k$ if

$$\tilde{v}_1(\mathbf{p}) = \tilde{v}_2(\mathbf{p}) \qquad \text{for} \qquad \|\mathbf{p}\| \leq 2k.$$

A stronger sufficient condition

Two dimensions: In the first Born approximation, the scattering amplitudes of the potentials v_1 and v_2 are indistinguishable at wave-numbers $k' < k$ if

$$\tilde{v}_1(p_x, p_y) = \tilde{v}_2(p_x, p_y) \qquad \text{for} \qquad |p_y| \leq 2k.$$



Example

$$v_1(x,y) = g(x)\delta(y) \qquad v_2(x,y) = g(x)\frac{\sin \beta y}{\pi y}$$

$$\tilde{v}_1(p_x,p_y) = \tilde{g}(p_x) \qquad \tilde{v}_2(p_x,p_y) = \tilde{g}(p_x)\chi_{[-\beta,\beta]}(p_y)$$

$\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]$, defined as:

$$\chi_{[a,b]}(x) := \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Consider a unit vector $\mathbf{u}$ and suppose that $\mathbf{p} = (\mathbf{p}_\perp, p_\shortparallel)$ where $p_\shortparallel := \mathbf{u}\cdot\mathbf{p}$, and $\mathbf{p}_\perp := \mathbf{p} - p_\shortparallel \mathbf{u}$. Let $\widehat{\mathscr{V}}_k := \widehat{\Pi}_k v(\hat{\mathbf{x}})\widehat{\Pi}_k$ in which

$$\widehat{\Pi}_k := \int_{S_\perp} d\mathbf{p}_\perp \int_{-k}^{k} dp_\shortparallel \,|\mathbf{p}_\perp, p_\shortparallel\rangle\langle\mathbf{p}_\perp, p_\shortparallel|\,,$$

and consequently

$$\left\langle \mathbf{p}|\widehat{\mathscr{V}}_k|\mathbf{q}\right\rangle = \begin{cases} w(\mathbf{p}-\mathbf{q}) & \text{if} \quad p_\shortparallel - q_\shortparallel \in [-2k, 2k] \\ 0 & \text{otherwise} \end{cases}$$

Lemma

The potential $v(\hat{\mathbf{x}})$ and $\widehat{\mathscr{V}}_k$ are indistinguishable at scattering processes with wave number $k' \leq k$ unless we go beyond the first Born approximation.

The propagating wave approximation

Definition

The propagating wave approximation is the approximation method, in which $\widehat{\mathscr{V}}_k$ replaces $v(\hat{x})$ in the Lippmann-Schwinger equation.

Remark

In this approximation, the scattering amplitude at $k' < k$ is identical for all potentials v with the same $\tilde{v}(\mathbf{p})$, $p_y \in [-2k, 2k]$.

Suppose

$$\widehat{\Pi}_k := \int_{\mathbb{R}^2} d^2p \int_{-k}^{k} dp_y \left|\mathbf{p}\right\rangle \left\langle\mathbf{p}\right| .$$

Solutions to the Lippmann-Schwinger equation

$$\left|\psi\right\rangle_{\mathrm{PWA}} = \left|\mathbf{k}\right\rangle + G(\hat{\mathbf{p}})\widehat{\mathscr{V}}_k \left|\psi\right\rangle_{\mathrm{PWA}} ,$$

are given by the Born series

$$\left\langle \mathbf{p}|\psi^{(n)}\right\rangle_{\mathrm{PWA}} = \chi_{[-k,k]}(p_y)\, G(\mathbf{p}) \int_{\mathbb{R}^2} d^2q \int_{-k}^{k} dq_y w(\mathbf{p}-\mathbf{q}) \left\langle \mathbf{q}|\psi^{(n-1)}\right\rangle_{\mathrm{PWA}}$$

$$f^{(n)}_{\mathrm{PWA}}(\mathbf{k}_s,\mathbf{k}) = -\frac{1}{4\pi}\int_{\mathbb{R}^2} d^2q \int_{-k}^{k} dq_y\, \tilde{v}(\mathbf{k}_s-\mathbf{q}) \left\langle \mathbf{q}|\psi^{(n-1)}\right\rangle_{\mathrm{PWA}} .$$

Remark

$\left|\psi^{(0)}\right\rangle_{\mathrm{PWA}} = |\mathbf{k}\rangle$ and consequently $f^{(1)} = f^{(1)}_{\mathrm{PWA}}$.

$$f(\mathbf{k}_s,\mathbf{k}) - f_{\mathrm{PWA}}(\mathbf{k}_s,\mathbf{k}) = \sum_{n=2}\left[f^{(n)}(\mathbf{k}_s,\mathbf{k}) - f^{(n)}_{\mathrm{PWA}}(\mathbf{k}_s,\mathbf{k})\right],$$

Second Born approximation

$$
\begin{aligned}
\delta f^{(2)}(\mathbf{k}_s,\mathbf{k}) &:= f^{(2)}(\mathbf{k}_s,\mathbf{k}) - f^{(2)}_{\mathrm{PWA}}(\mathbf{k}_s,\mathbf{k}) \\
&= \frac{1}{8\pi^3}\int_{\mathbb{R}^2}^{\infty} d^2q \int_{\mathbb{R}\setminus[-k,k]} dq_y\, \tilde{v}(\mathbf{k}_s-\mathbf{q})\, G(\mathbf{q})\, \tilde{v}(\mathbf{q}-\mathbf{k}) \\
&= -\frac{1}{4\pi}\int dy \int d\mathbf{x}_1 \int d\mathbf{x}_2 \Bigg\{ v(\mathbf{x}_1)\, v(\mathbf{x}_2)\, e^{-i\mathbf{k}_s\cdot\mathbf{x}_1} e^{i\mathbf{k}\cdot\mathbf{x}_2} \\
&\times \frac{e^{ik\|y\mathbf{e}_y+\mathbf{x}_2-\mathbf{x}_1\|}}{\|y\mathbf{e}_y+\mathbf{x}_2-\mathbf{x}_1\|}\left[\delta(y) - \frac{\sin ky}{\pi y}\right]\Bigg\}.
\end{aligned}
$$

where $\mathbf{e}_y$ denotes the unit vector along the y direction.

Potentials with exact propagating wave approximation

$$\tilde{v}(\mathbf{p}) = 0 \quad \text{for} \quad p_y \leq 0.$$

In one dimension, scattering potentials with this property are known to be unidirectionally invisible for all wavenumbers.

1. S. A. R. Horsley, M. Artoni and G. C. La Rocca, Spatial Kramers-Kronig relations and the reflection of waves, Nature Photonics 9, 436-439 (2015).
2. S. Longhi, Wave reflection in dielectric media obeying spatial Kramers-Kronig relations, EPL 112, 64001 (2015).
3. S. A. R. Horsley and S. Longhi, One-way invisibility in isotropic dielectric optical media, Amer. J. Phys. 85, 439-446 (2017).
4. W. Jiang, Y. Ma, J. Yuan, G. Yin, W. Wu, and S. He, Deformable broadband metamaterial absorbers engineered with an analytical spatial Kramers-Kronig permittivity profile, Laser Photonics Rev. 11, 1600253 (2017).

For such potentials the Born series is given by

$$\left\langle \mathbf{p} | \psi^{(n)} \right\rangle = G(\mathbf{p}) \int_{\mathbb{R}^2} d^2 q \int_{-\infty}^{p_y} dq_y \, w(\mathbf{p} - \mathbf{q}) \left\langle \mathbf{q} | \psi^{(n-1)} \right\rangle .$$

Lemma

$$\langle \mathbf{p} | \psi \rangle \quad \textit{for} \quad p_y < -k.$$

Proof.

For $p_y < -k$, $\left\langle \mathbf{p} | \psi^{(0)} \right\rangle = \langle \mathbf{p} | \mathbf{k} \rangle = 0$. Assuming $\left\langle \mathbf{p} | \psi^{(n-1)} \right\rangle = 0$, the iterative formula gives $\left\langle \mathbf{p} | \psi^{(n)} \right\rangle = 0$. □

The iterative formula is given by

$$\left\langle \mathbf{p} | \psi^{(n)} \right\rangle = \theta(p_y + k) G(\mathbf{p}) \int_{\mathbb{R}^2} d^2 q \int_{-k}^{p_y} dq_y \, w(\mathbf{p} - \mathbf{q}) \left\langle \mathbf{q} | \psi^{(n-1)} \right\rangle ,$$

where θ denote the Heaviside function, i.e., $\theta(x)$ equals 0 and 1, for $x < 0$ and $x > 0$ respectively.

Comparing this result with

$$\left\langle \mathbf{p} | \psi^{(n)} \right\rangle_{\mathrm{PWA}} = \chi_{[-k,k]}(p_y)\, G(\mathbf{p}) \int_{\mathbb{R}^2} d^2q \int_{-k}^{k} dq_y w(\mathbf{p}-\mathbf{q}) \left\langle \mathbf{q} | \psi^{(n-1)} \right\rangle_{\mathrm{PWA}}$$

we confirm the following theorem.

Theorem

The scattering amplitude computed in the propagating wave approximation is exact for potentials v if $\tilde{v}(\mathbf{p}) = 0$ for $p_y \leq 0$.

Thank you!