RDP School and Workshop on Mathematical Physics

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Split octonionic field theories

PhD student: **Alexandre Gurchumelia**

Supervisor: professor **Merab Gogberashvili**

Ivane Javakhishvili Tbilisi State University
Evgeni Kharadze National Astrophysical Observatory

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Outline

- Hypercomplex algebras and split octonions $\mathbb{O}'$
- Triality in $(4+4)$-space
- Hypercomplex analysis for $\mathbb{O}' \to \mathbb{O}'$ functions
- Constructing Lagrangians
- Dirac and dyonic Maxwell theories with $\mathbb{O}'$

Octonions in physics

Octonions

- Color symmetry[Gunaydin, Gursey 1973; Morita 1981], Quantum mechanics[Gunaydin, Piron, Ruegg 1978], GUT[Sudbery 1984, Dixon 1990; Castro 2007], wave equations[Kurdgelaidze 1985, Gogberashvili 2006], associator quantization[Lohmus, Paal, Sorgsepp 1998], M theory[Lukierski, Toppan 2002], quantum Hall effect[Bernevig, Hu, *et al* 2003]

Split octonions

- Particle generations[Gunaydin, Gursey 1974, Silagadze 1995], electrodynamics[Nash 1989], gravity[Nash 2010], geometry[Gogberashvili 2009, 2015].

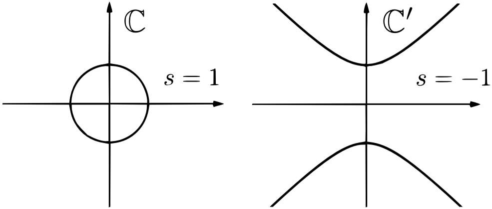
Cayley-Dickson constructions

Given an involution algebra $\mathbb{A}$ we construct $\mathbb{A} \oplus \mathbb{A}$ where

$$\begin{aligned} \text{invol:} \quad & x^* = (x_1, x_2)^* = (x_1^*, -x_2) \\ \text{multip:} \quad & x\, y = (x_1,\, x_2)\,(y_1,\, y_2) = (x_1 y_1 \pm \overline{y}_2 x_2,\; y_2 x_1 + x_2 \overline{y}_1) \end{aligned} \tag{1}$$

We get $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \cdots$ or $\mathbb{R} \to \mathbb{C}' \to \mathbb{H}' \to \mathbb{O}' \to \cdots$

$$\begin{aligned} \text{for } s = 1 \text{ we have} \quad & i^2 = -1 \quad \to \quad e^{i\varphi} = \cos\varphi + i \sin\varphi \\ \text{for } s = -1 \text{ we have} \quad & j^2 = 1 \quad \to \quad e^{j\varphi} = \cosh\varphi + j \sinh\varphi \end{aligned} \tag{2}$$



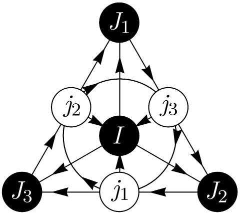
Properties of Cayley–Dickson algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$
dimension: $\mathbb{R}^n$	1	2	4	8	16
order: $a < b$	✓	×	×	×	×
commutative: $ab = ba$	✓	✓	×	×	×
associative: $a\,(bc) = (ab)\,c$	✓	✓	✓	×	×
alternative: $(xx)\,y = x\,(xy)$	✓	✓	✓	✓	×

First four algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only four normed division algebras (Hurwitz algebras).

General element $x \in \mathbb{O}'$ and its conjugate

$$
\begin{aligned}
x &= x_0 + I x_4 + \sum_{1,2,3} \left(j_n x_n + J_n x_{4+n} \right) \\
\overline{x} &= x_0 - I x_4 - \sum_{1,2,3} \left(j_n x_n + J_n x_{4+n} \right)
\end{aligned}
\tag{3}
$$

properties	Algebraic relations	Fano plane
alternativity	$j_m j_n = -\delta_{mn} + \sum_k \epsilon_{mnk} j_k$,	
$(xx)\,y = x\,(xy)$	$I^2 = 1$,	
$(xy)\,y = x\,(yy)$	$J_m J_n = \delta_{mn} + \sum_k \epsilon_{mnk} j_k$,	
$\Downarrow$		
flexibility	$j_n I = J_n$,	
$(xy)\,x = x\,(yx)$	$J_m j_n = \delta_{mn} I - \sum_k \epsilon_{mnk} J_k$.	

Symbolic computations with SplitOct python library

SplitOct_arithmetics.ipynb

```python
[1]: from SplitOct import *
```

```python
[2]: x = (5 + 3*j2 - J3) * (j1 + J2 - J3)
     display(x)
```

$$1 + 6j_1 - 3j_3 - 3I + 3J_1 + 6J_2 - 5J_3$$

```python
[3]: y = x.conj()
     display(y)
```

$$1 - 6j_1 + 3j_3 + 3I - 3J_1 - 6J_2 + 5J_3$$

```python
[4]: x * y
```

[4]: -33

Gurchumelia, A. (2023). SplitOct [Computer software]
github.com/EQUINOX24/SplitOct

Symmetric nondegenerate bilinear form $\langle\cdot,\cdot\rangle : \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ is just an *"inner product"* in $(4+4)$-space

$$\begin{aligned} \langle x,\, y\rangle &= \frac{1}{2}\left(\overline{x}\, y + \overline{y}\, x\right) \\ &= \sum_{1,2,3}\left(x_n\, y_n - x_{4+n}\, y_{4+n}\right) \end{aligned} \tag{4}$$

Quadratic from $\mathcal{Q} : \mathbb{O}' \to \mathbb{R}$

$$\mathcal{Q}\,(x) = \langle x,\, x\rangle \tag{5}$$

Trilinear form $\mathcal{F} : \mathbb{O}' \times \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$

$$\mathcal{F}\,(\phi, \chi, \psi) = \langle\, \overline{\phi},\, \chi\, \psi\rangle \tag{6}$$

(4,4)-space pseudo-orthogonal groups with $\mathbb{O}'$

$SO\left(4,4\right)$ vector $\chi \in \mathbb{O}'$ and $Spin\left(4,4\right)$ spinor chiral parts $\phi, \psi \in \mathbb{O}'$ transform as

$$\begin{cases} \phi' = u^2 \left(\phi u\right) T_{uv}\left(s_u \vartheta\right) \ , \\ \chi' = T_{uv}\left(\vartheta\right)\left(u \chi u\right) T_{uv}\left(\vartheta\right), \\ \psi' = u^2 T_{uv}\left(s_u \vartheta\right)\left(u \psi\right) \ , \end{cases} \tag{7}$$

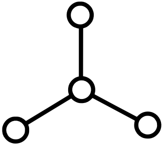
where $u, v \in \text{basis}\left(\mathbb{O}'\right)$ and

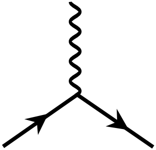
$$T_{uv}\left(\vartheta\right) = \begin{cases} u \cos \frac{\vartheta}{2} + v \sin \frac{\vartheta}{2}, & u\overline{u} = v\overline{v} \\ u \cosh \frac{\vartheta}{2} + v \sinh \frac{\vartheta}{2}, & u\overline{u} = -v\overline{v} \end{cases} \tag{8}$$

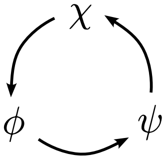
$$s_u = \left| u\overline{u} - u^2 \right| - 1 = \pm 1 \tag{9}$$

Triality of 8 dimensional space

Triality symmetry is usually defined in terms of Dynkin diagram D_4 of 8 dimensional orthogonal groups. J. Baez has pointed out possible connection fermion/boson/fermion Feynman diagram







Triality in $\mathbb{O}'$: `Gogberashvili, Gurchumelia (2023)` & `Mikosz (2013)`
Triality in $\mathbb{O}$: `Gamba (1967), Peculiarities of the Eight-Dimensional Space`
Supersym D_4: `Frank, Smith (1993)` & `Baez (2001) Spinors and Trialities`

Manifestation of triality in the transformations

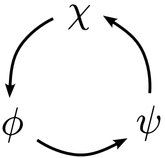
In tangential space $T_{1j_1}(\vartheta)$ gives

$$\begin{cases} \phi'_0 = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\ \phi'_1 = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\ \phi'_2 = \phi_2 - \frac{1}{2}\vartheta\phi_3 \\ \phi'_3 = \phi_3 + \frac{1}{2}\vartheta\phi_2 \\ \phi'_4 = \phi_4 - \frac{1}{2}\vartheta\phi_5 \\ \phi'_5 = \phi_5 + \frac{1}{2}\vartheta\phi_4 \\ \phi'_6 = \phi_6 + \frac{1}{2}\vartheta\phi_7 \\ \phi'_7 = \phi_7 - \frac{1}{2}\vartheta\phi_6 \end{cases} \begin{cases} \chi'_0 = \chi_0 - \vartheta\chi_1 \\ \chi'_1 = \chi_1 + \vartheta\chi_0 \\ \chi'_2 = \chi_2 \\ \chi'_3 = \chi_3 \\ \chi'_4 = \chi_4 \\ \chi'_5 = \chi_5 \\ \chi'_6 = \chi_6 \\ \chi'_7 = \chi_7 \end{cases} \begin{cases} \psi'_0 = \psi_0 + \frac{1}{2}\vartheta\psi_1 \\ \psi'_1 = \psi_1 - \frac{1}{2}\vartheta\psi_0 \\ \psi'_2 = \psi_2 + \frac{1}{2}\vartheta\psi_3 \\ \psi'_3 = \psi_3 - \frac{1}{2}\vartheta\psi_2 \\ \psi'_4 = \psi_4 + \frac{1}{2}\vartheta\psi_5 \\ \psi'_5 = \psi_5 - \frac{1}{2}\vartheta\psi_4 \\ \psi'_6 = \psi_6 - \frac{1}{2}\vartheta\psi_7 \\ \psi'_7 = \psi_7 + \frac{1}{2}\vartheta\psi_6 \end{cases} \tag{10}$$

If we compose transformations $T_{j_1 1}\left(\frac{\vartheta}{2}\right)$, $T_{j_2 j_3}\left(\frac{\vartheta}{2}\right)$, $T_{J_1 I}\left(\frac{\vartheta}{2}\right)$, $T_{J_2 J_3}\left(\frac{\vartheta}{2}\right)$ we'll roles of ϕ, χ and ψ *trially* swap

Manifestation of triality in the transformations

$$
\begin{cases}
\phi'_0 = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\
\phi'_1 = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\
\phi'_2 = \phi_2 + \frac{1}{2}\vartheta\phi_3 \\
\phi'_3 = \phi_3 - \frac{1}{2}\vartheta\phi_2 \\
\phi'_4 = \phi_4 + \frac{1}{2}\vartheta\phi_5 \\
\phi'_5 = \phi_5 - \frac{1}{2}\vartheta\phi_4 \\
\phi'_6 = \phi_6 - \frac{1}{2}\vartheta\phi_7 \\
\phi'_7 = \phi_7 + \frac{1}{2}\vartheta\phi_6
\end{cases}
\quad
\begin{cases}
\chi'_0 = \chi_0 + \frac{1}{2}\vartheta\chi_1 \\
\chi'_1 = \chi_1 - \frac{1}{2}\vartheta\chi_0 \\
\chi'_2 = \chi_2 - \frac{1}{2}\vartheta\chi_3 \\
\chi'_3 = \chi_3 + \frac{1}{2}\vartheta\chi_2 \\
\chi'_4 = \chi_4 - \frac{1}{2}\vartheta\chi_5 \\
\chi'_5 = \chi_5 + \frac{1}{2}\vartheta\chi_4 \\
\chi'_6 = \chi_6 + \frac{1}{2}\vartheta\chi_7 \\
\chi'_7 = \chi_7 - \frac{1}{2}\vartheta\chi_6
\end{cases}
\quad
\begin{cases}
\psi'_0 = \psi_0 - \vartheta\psi_1 \\
\psi'_1 = \psi_1 + \vartheta\psi_0 \\
\psi'_2 = \psi_2 \\
\psi'_3 = \psi_3 \\
\psi'_4 = \psi_4 \\
\psi'_5 = \psi_5 \\
\psi'_6 = \psi_6 \\
\psi'_7 = \psi_7
\end{cases}
. \qquad (11)
$$



Analysis of $\mathbb{O}' \to \mathbb{O}'$ functions

Derivatives are

$$\begin{aligned} \partial &= \tfrac{1}{2}\left(\partial_0 + I\partial_4\right) + \tfrac{1}{2}\sum_n \left(j_n\partial_n + J_n\partial_{4+n}\right) \\ \overline{\partial} &= \tfrac{1}{2}\left(\partial_0 - I\partial_4\right) - \tfrac{1}{2}\sum_n \left(j_n\partial_n + J_n\partial_{4+n}\right) \end{aligned} \tag{12}$$

But usual properties hold only for linear functions

$$\partial x = \overline{\partial}\,\overline{x} = 1\,, \qquad \overline{\partial} x = \partial \overline{x} = 0\,. \tag{13}$$

Generalizaiton of Cauchy-Riemann-Fueter equations to $f : \mathbb{O}' \to \mathbb{O}'$

$$\begin{aligned} \partial f &= 0 \quad \text{analyticity} \\ \overline{\partial} f &= 0 \quad \text{anti-analyticity} \end{aligned} \tag{14}$$

For $\mathbb{H}$: `DeLeo (2003) Quaternionic analysis. Appl. Math. Letters`
For $\mathbb{O}$: `Kauhanen, Orelma (2018) Cauchy-Riemann Operators in Octonionic Analysis. Adv. Appl. Cliff. Algebras`

Constructing a Lagrangian

In $\mathcal{F}(\phi, \chi, \psi) = \langle \overline{\phi}, \chi\psi \rangle$ we replace $\chi \to \overrightarrow{\partial}$ and set ϕ and ψ to be functions of $x \in \mathbb{O}'$. Stationarizing $S = \int d^8x \, \mathcal{L}$ with Lagrangian

$$\mathcal{L} = \left\langle \overline{\phi}, \overrightarrow{\partial}\psi \right\rangle + \frac{1}{2}\lambda_1 \langle \phi, \phi \rangle + \frac{1}{2}\lambda_2 \langle \psi, \psi \rangle \tag{15}$$

gives two equations

$$\begin{cases} \overline{\overrightarrow{\partial}}\,\overline{\phi} = \lambda_2 \psi \\ \overrightarrow{\partial}\psi = -\lambda_1 \phi \end{cases} \tag{16}$$

If we set $\lambda_2 = 0$ we get eight independent equations

$$\left\langle \overrightarrow{\partial}, \overrightarrow{\partial} \right\rangle \psi = 0 \tag{17}$$

where $\left\langle \overrightarrow{\partial}, \overrightarrow{\partial} \right\rangle$ is d'Alembert like operator in (4+4) space.

Free Maxwell theory

In the above Lagrangian we set $\lambda_2 = 0$ and $\lambda_1 = -1$ and replace

$$\begin{aligned}
\partial &\to D = I\partial I = \frac{1}{2}\left(\partial_0 + I\partial_4\right) - \frac{1}{2}\sum_n \left(j_n\partial_n + J_n\partial_{4+n}\right) \\
\psi &\to A = \mathcal{C}_0 + I\mathcal{A}_0 + \frac{1}{2}\sum_n \left(J_n\mathcal{C}_n + j_n\mathcal{A}_n\right) \\
\phi &\to \overline{F} \qquad \text{where } F = \overrightarrow{D}A
\end{aligned} \tag{18}$$

$$\text{to get} \qquad \mathcal{L} = \frac{1}{4}\langle F, F\rangle \qquad \to \qquad \left\langle \overrightarrow{D}, \overrightarrow{D} \right\rangle A = 0. \tag{19}$$

($\mathcal{A}$ - EM 4-potential, $\mathcal{C}$ - magnetic monopole 4-potential)
If we get rid of extra dimension $\mathscr{D} = \frac{1}{2}\left(I\partial_t - j_1\partial_x - j_2\partial_y - j_3\partial_z\right)$ the above equation reduces to free dyonic Maxwell equation

$$\left\langle \overrightarrow{\mathscr{D}}, \overrightarrow{\mathscr{D}} \right\rangle A = 0\,. \tag{20}$$

Free Dirac theory

Second EoM of the following Lagrangian

$$\mathcal{L} = \left\langle \overline{\phi}, \overrightarrow{D}\psi \right\rangle - \frac{1}{2} m \left\langle \overline{\phi}, J_3\psi \right\rangle \qquad \rightarrow \qquad \begin{cases} \left(\overrightarrow{D} - \frac{1}{2} J_3 m \right) \overline{\phi} = 0 \\ \left(\overrightarrow{D} - \frac{1}{2} J_3 m \right) \psi = 0 \end{cases} \tag{21}$$

in the same limit when extra dimensions are ignored $D \rightarrow \mathscr{D}$ reduces to free Dirac equation

$$\overrightarrow{\mathscr{D}}\psi = \frac{1}{2} J_3 m \psi \tag{22}$$

Lagrangian could also be constructed without second ϕ field

$$\mathcal{L} = \frac{1}{2} \left\langle J_3\psi, \left(\mathscr{D} - mJ_3\right)\psi \right\rangle \qquad \rightarrow \qquad \left(\overrightarrow{\mathscr{D}} - \frac{1}{2} J_3 m \right) \psi \tag{23}$$

Dirac equation system in external potential

Equation (no Lagrangian yet) is

$$\left(\overrightarrow{\mathscr{D}} - \frac{1}{2} J_3 m\right) \psi = J_3 \left(\mathrm{conj}_{Ij}\left(A\psi\right) I\right) \tag{24}$$

where

$$\mathrm{conj}_{Ij}\left(x\right) = x_0 - I x_4 + \sum_n \left(-j_n x_n + J_n x_{4+n}\right) . \tag{25}$$

Other possible terms in Lagrangian

We could add non $SO\,(4,4)$-$Spin\,(4,4)$ invariant terms like $\mathrm{Re}\,(ABC\ldots)$ to the Lagrangian where $A, B, C \in \mathbb{O}'$, symmetry group under which these kind of terms would be invariant is $\mathbb{O}'$ automorphism group called non-compact G_2.

We've studied Casimir operator of this exceptional semi-simple Lie group and showed that in the limit when extra dimensions are kept constant it reduces to a sum of Poincare $(P_\mu P^\mu)$ and Lorentz group $(L_n L^n - K_n K^n)$ Casimir operators.

G_2 casimir: `Gogberashvili & Gurchumelia (2019)`

In summary

- We constructed Lagrangian using trilinear form on trial ϕ, ψ chiral spinors and $\chi \to \partial$ vector that give system of left pseudo-analyticity and pseudo-antianalyticity equations

$$\begin{cases} \overrightarrow{\overline{\partial}}\,\overline{\phi} = \lambda_2 \psi \\ \overrightarrow{\partial}\,\psi = -\lambda_1 \phi \end{cases}$$

- Upon altering the gradient operator $\partial \to I\partial I$ these equations become generalizations of free Dirac and dyonic Maxwell systems with 1 extra spatial and and 3 extra time dimensions.
- Triality symmetry could help interacting theory that exhibits SUSY-like properties

Publications

- Gogberashvili, M., & Gurchumelia, A. (2023). Dirac and Maxwell systems in split octonions. Journal of Applied Mathematics and Physics 11
- Gurchumelia, A., & Gogberashvili, M. (2021, April). Split Octonions and Triality in (4+4)-Space. In RDP online workshop" Recent Advances in Mathematical Physics (p. 8).
- Gogberashvili, M., & Gurchumelia, A. (2019). Geometry of the non-compact G(2). Journal of Geometry and Physics, 144, 308-313.
- Gogberashvili, Merab, and Otari Sakhelashvili. "Geometrical applications of split octonions." Advances in Mathematical Physics 2015 (2015).
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- Gogberashvili, M. (2006). Octonionic electrodynamics. Journal of Physics A: Mathematical and General, 39(22), 7099.
- Gogberashvili, M. (2002). Observable algebra. arXiv preprint hep-th/0212251.