Grad–Shafranov equation in cap-cyclide coordinates: general Heun function Solution

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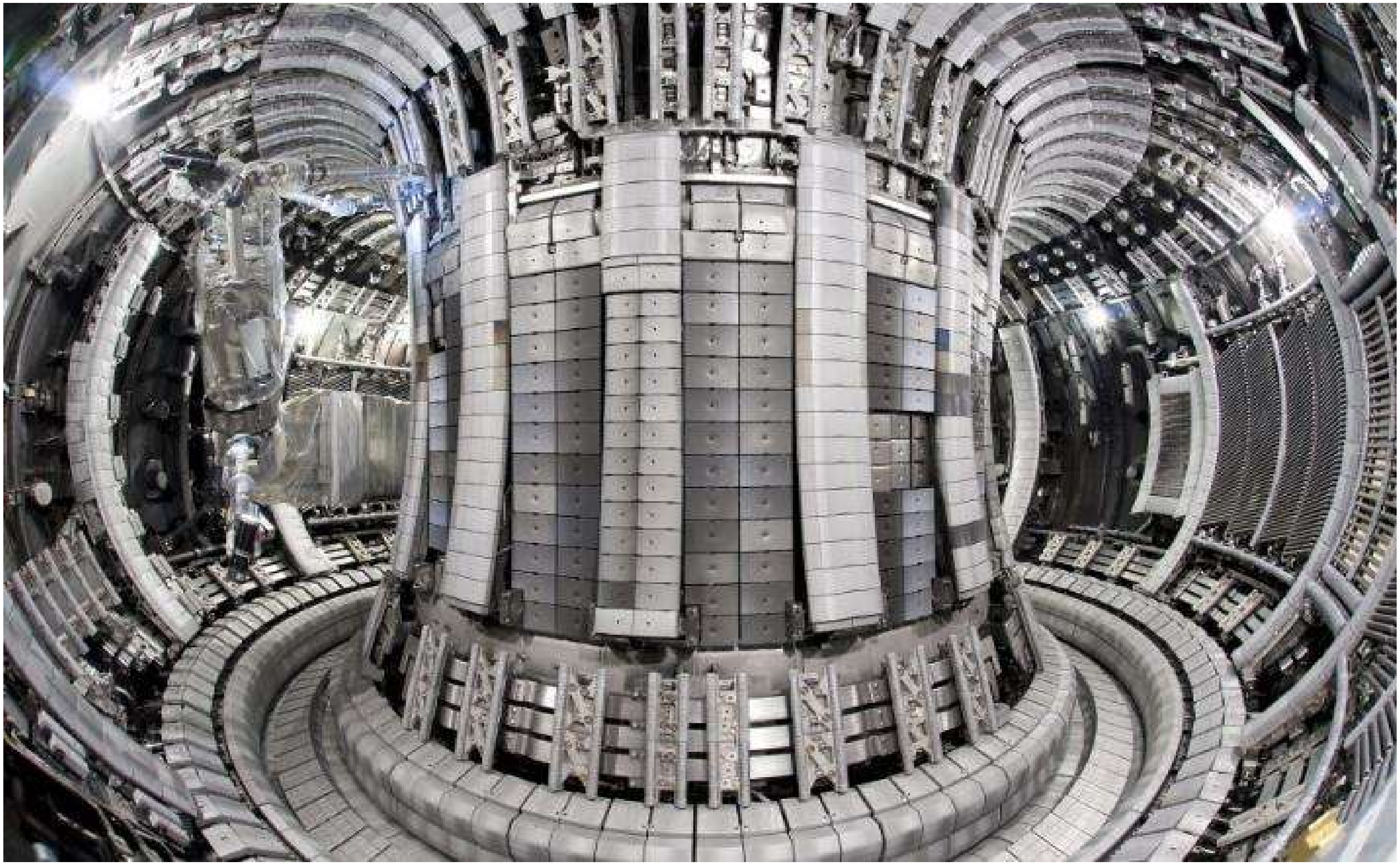
Outline

- Tokamak (thermonuclear reactor) plasma shaping
- Cap-Cyclide coordinates
- Grad–Shafranov equation
  Generalized Laplace equation
- Solution in Bipolar coordinates
- Solution in Cap-Cyclide coordinates
  General Heun equation
- Bipolar limit of Cap-Cyclide coordinates for GSE
- Discussion

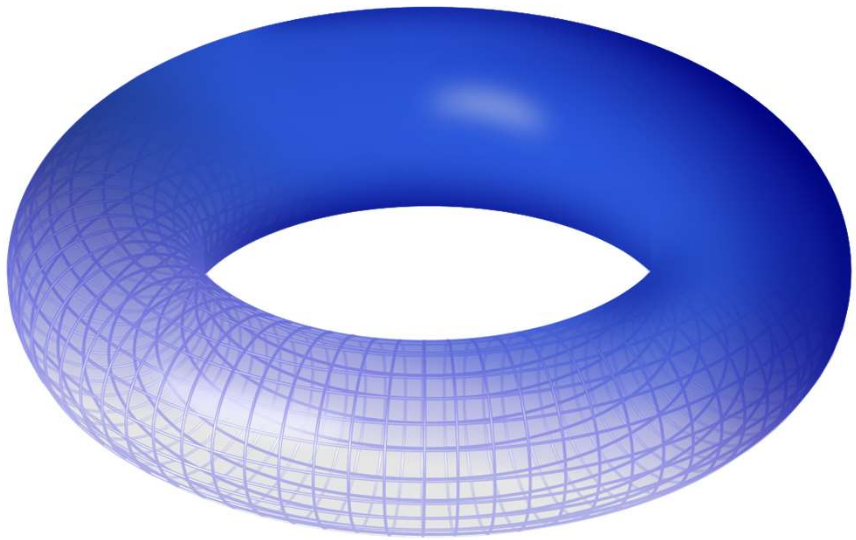
Plasma - Tokamak

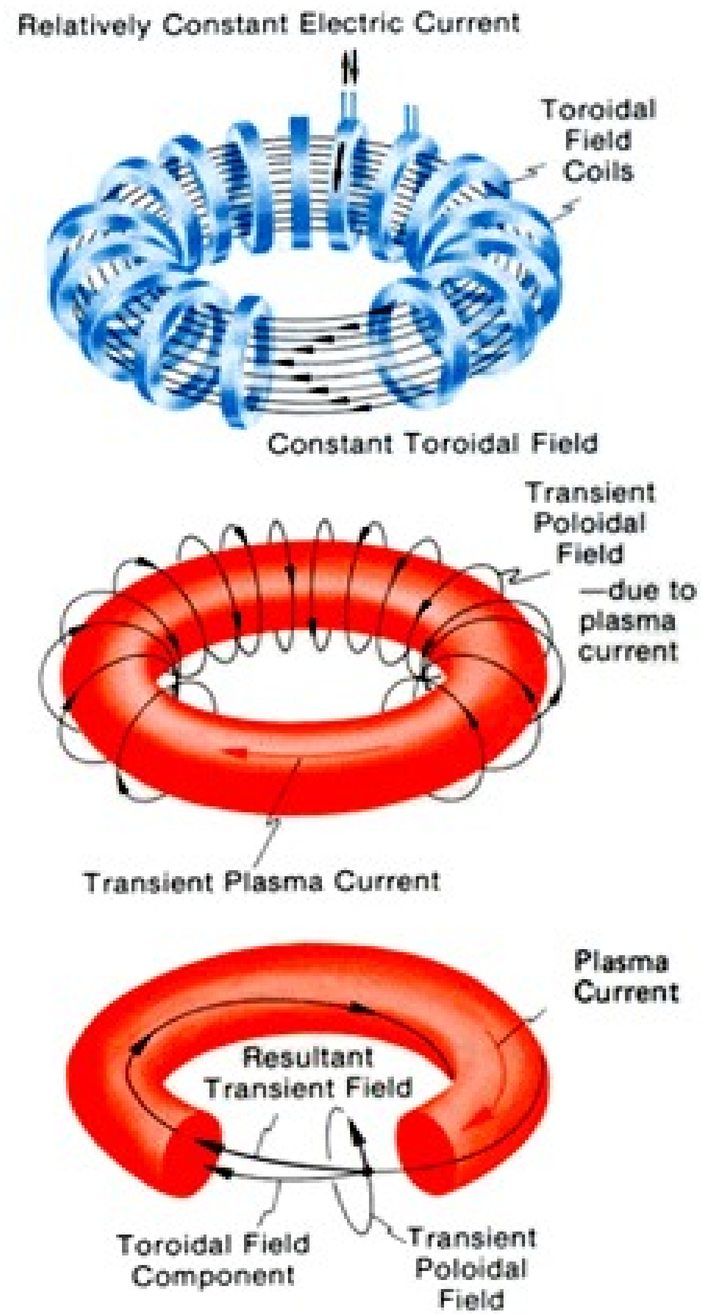


A Tokamak is a device which uses a magnetic field to confine plasma in the shape of a torus

It is designed to produce controlled fusion energy. It is regarded as the main candidate for the role of a practical thermonuclear reactor

Initially: Torus configuration





A stable plasma equilibrium requires magnetic field lines that spiral around the torus.

Lawson Criterion – triple product

Density x Temperature x Confinement time:

$$nT\tau$$

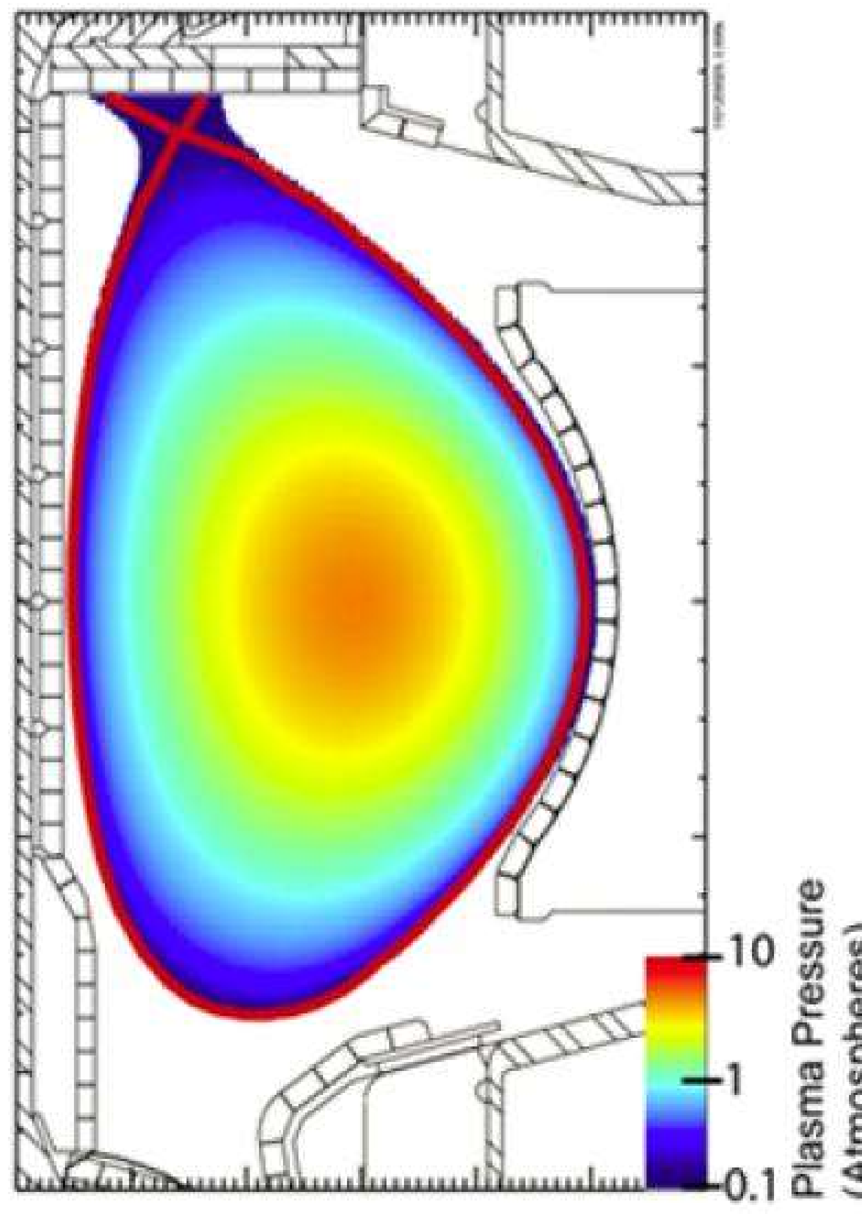
Plasma shaping - Geometric factors

Geometric factors influencing energy confinement time:

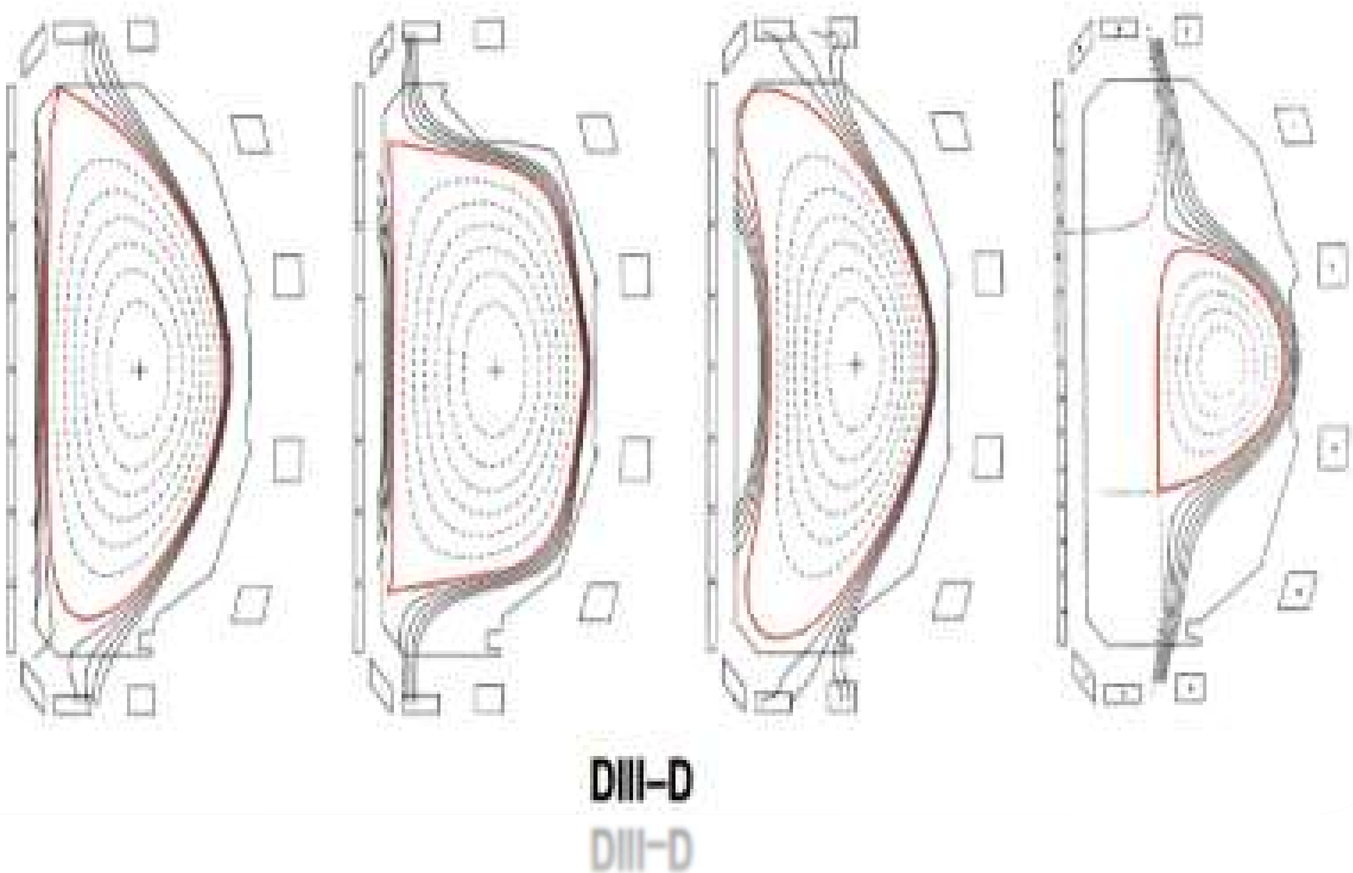
- Aspect ratio, $\Lambda = R / a$

- Plasma elongation, $k=b/a$, where b is the height of the plasma measured from the equatorial plane

- Plasma triangularity, δ, the horizontal distance between the major radius and the *x*-point



Plasma Shaping

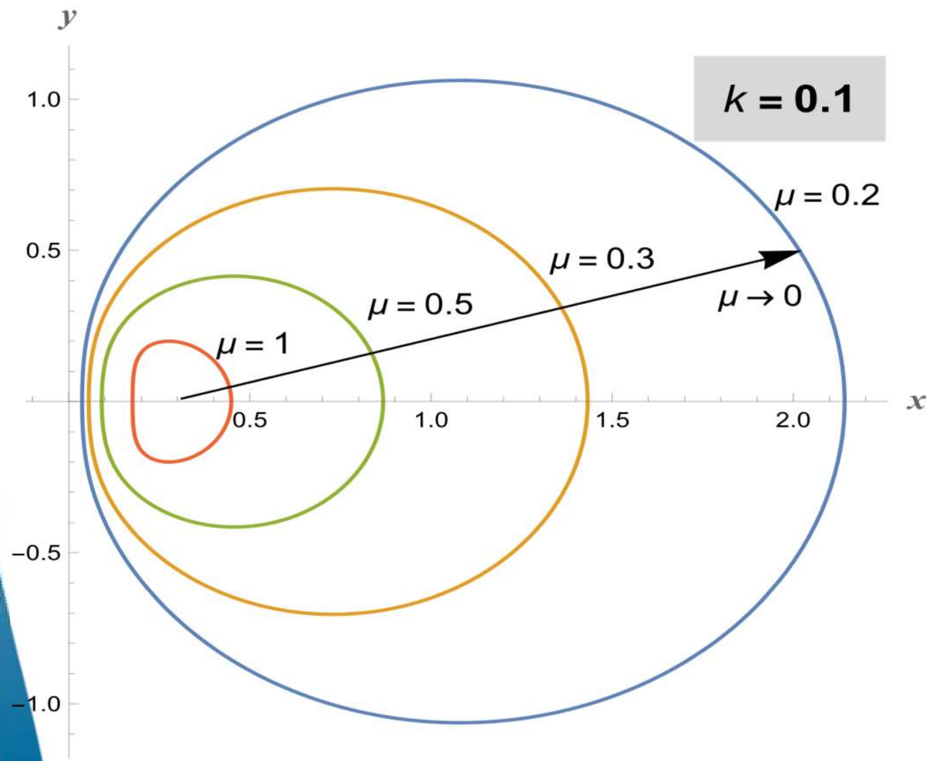


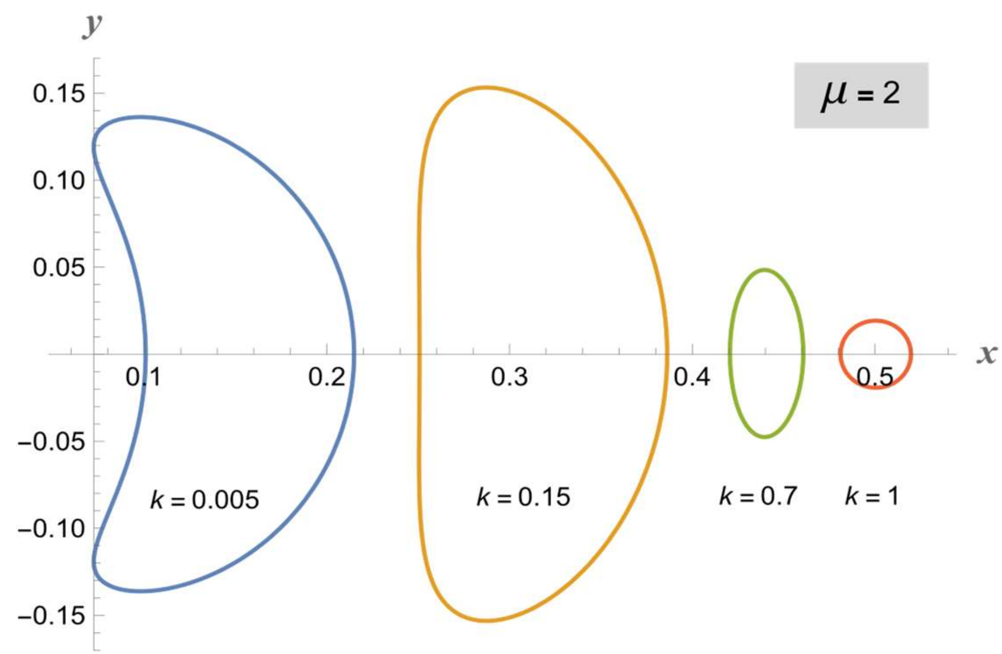
- Premier plasma shaping tokamak
- Modeled confinement time in excess of 300s
- Internal transport barrier optimization

➢ Cap-cyclide coordinates

A natural coordinate system

Cap-cyclide coordinates





Cap-cyclide coordinates

Real space coordinates (x,y,z)
are transformed to the coordinates (μ,ν,φ)

$$x = \frac{\Lambda}{a\Gamma} sn(\mu, k) dn(\nu, k_1) \cos\varphi$$

$$y = \frac{\Lambda}{a\Gamma} sn(\mu, k) dn(\nu, k_1) \sin\varphi$$

$$z = \frac{k^{1/4}\Pi}{2a\Gamma} \qquad k + k_1 = 1$$

Involved functions

$$\Lambda = 1 - dn^2(\mu,k)\,sn^2(\nu,k_1)$$

$$\Gamma = sn^2(\mu,k)\,dn^2(\nu,k_1) + \\ + \left(\Lambda / k^{1/4} + cn(\mu,k)\,dn(\mu,k)\,sn(\nu,k_1)\,cn(\nu,k_1)\right)^2$$

$$\Pi = \left(\Lambda^2 / k^{1/2}\right) - \left(sn^2(\mu,k)\,dn^2(\nu,k_1) + \\ + cn^2(\mu,k)\,dn^2(\mu,k)\,sn^2(\nu,k_1)\,cn^2(\nu,k_1)\right)$$

k is the parameter of elliptical integrals

$$k + k_1 = 1$$

k_1 is the complementary parameter of elliptical integrals

Complex function representation

Coordinate transformation: $(R,Z)\rightarrow(\mu,\nu)$

$$\frac{k^{1/4}}{2ai}\cdot\frac{1+ik^{1/4}sn(w)}{1-ik^{1/4}sn(w)}=R(\mu,\nu)+iZ(\mu,\nu)$$

$$w(\mu,\nu)=\mu+i\nu$$

Cauchy–Riemann conditions

$$\frac{\partial R}{\partial\mu}=\frac{\partial Z}{\partial\nu}\qquad\frac{\partial R}{\partial\nu}=-\frac{\partial Z}{\partial\mu}$$

- **Grad–Shafranov equation Generalized Laplace equation**

Plasma equilibrium equation

$$\nabla P = \mathbf{J} \times \mathbf{B}$$

P is the kinetic plasma pressure
J is the plasma current density
B is the magnetic field

Flux function over a poloidal surface

$$\psi = \frac{1}{2\pi} \iint_{S\text{pol}} \mathbf{B} \cdot d\mathbf{s}$$

Assuming axial symmetry and using cylindrical coordinates *R,Z,φ*

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = \mu_0 R J_\varphi$$

J_φ is the axisymmetric current density

This is the Grad–Shafranov equation

Grad-Shafranov and Laplace equations

$$\frac{\partial^2\psi}{\partial R^2} \mp \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = 0$$

Transformation $\psi = R^{-\sigma_0/2} w$

Auxiliary equation $\frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{\partial Z^2} + \frac{A}{R^2} = 0$

Axisymmetric Laplace equation

$\sigma_0 = +1$

$A = 1/4$

Grad-Shafranov equation

$\sigma_0 = -1, 3$

$A = -3/4$

Generalized Laplace equation

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{\sigma_0}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = 0$$

Laplacian

$$\Delta w(R,Z) = \frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{\partial Z^2}$$

$$R = \varphi_1(q_1,q_2) \qquad Z = \varphi_2(q_1,q_2)$$

$$\Delta w(q_1,q_2) = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_1}\frac{\partial w}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2}\frac{\partial w}{\partial q_2}\right)\right]$$

Scale factors (Lamé coefficients)

$$H_i = \sqrt{\left(\frac{\partial \varphi_1}{\partial q_1}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_2}\right)^2} \qquad i = 1,2$$

- **Solution in Bipolar coordinates**

2D *bipolar* coordinates r, θ

$$(x, y) = \left(\frac{a \sinh r}{\cosh r - \cos\theta}, \frac{a \sin\theta}{\cosh r - \cos\theta} \right)$$

2D Cartesian Laplacian

$$\Delta_C w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$(R, Z) = (x, y): \ \Delta_B w = \frac{(\cosh r - \cos\theta)^2}{a^2} \left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} \right)$$

GSE is reduced to the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{A w}{\sinh^2 r} = 0$$

Separation of variables

$$w = u(r)\ \Theta(\theta)$$

Separation constant p

$$\Theta = \sin(p\theta + \theta_0)$$

Equation for the radial component

$$\frac{d^2u}{dr^2} + \left(\frac{A}{\sinh^2 r} - p^2\right) u = 0$$

General solution

$$u(r)=\sqrt{\sinh r}\left(C_1 P^q_{p-1/2}\left(\cosh r\right)+C_2 Q^q_{p-1/2}\left(\cosh r\right)\right)$$

$$q=\sqrt{\frac{1}{4}-A}$$

For axisymmetric Laplace equation $q=0$

For Grad–Shafranov equation $q=1$

➢ Solution in Cap-Cyclide coordinates

Laplacian in cap-cyclide coordinates

$$\Delta_{CC} w = \frac{a^2\Gamma^2}{\Lambda^2\Omega^2}\left(\frac{\partial^2 w}{\partial \mu^2} + \frac{\partial^2 w}{\partial \nu^2}\right)$$

$$\Omega = \sqrt{\left(1 - sn(\mu,k)^2\, dn(\nu,k_1)^2\right)\left(dn(\nu,k_1)^2 - k\, sn(\mu,k)^2\right)}$$

Auxiliary equation is rewritten as

$$\frac{\partial^2 w}{\partial \mu^2} + \frac{\partial^2 w}{\partial \nu^2} + A\left(\frac{1}{sn(\mu,k)^2} + k\, sn(\mu,k)^2\right) -$$

$$- A\left(\frac{k}{dn(\nu,k_1)^2} + dn(\nu,k_1)^2\right) = 0$$

Separation of variables

$$w(\mu,\nu) = U(\mu)V(\nu)$$

$$U''(\mu) + A\left(\frac{1}{sn(\mu,k)^2} + k\,sn(\mu,k)^2 + B\right)U(\mu) = 0$$

$$V''(\nu) - A\left(\frac{k}{dn(\nu,k_1)^2} + dn(\nu,k_1)^2 + B\right)V(\nu) = 0$$

Variable change

$$U = z^{\sigma} y(z) \quad z = sn(\mu, k)^2$$

$$V = z^{\sigma} y(z) \quad z = dn(\nu, k_1)^2$$

$$\sigma = \frac{1}{4}\left(1 \pm \sqrt{1-4A}\right) = \frac{\sigma_0}{4}$$

General Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

$$(\gamma, \delta, \varepsilon, \alpha, \beta) = \left(\frac{1+4\sigma}{2}, \frac{1}{2}, \frac{1}{2}, 2\sigma, \frac{1}{2}\right)$$

$$\text{for } U: \quad a = \frac{1}{k}, \quad q = \frac{2(1+k)\sigma - A(B+k+1)}{4k}$$

$$\text{for } V: \quad a = k, \quad q = \frac{2(1+k)\sigma - A(B+k+1)}{4}$$

The five Heun equations_1

- Equations of the Heun class

$$(p_0 + p_1 z + p_2 z^2 + p_3 z^3)\frac{d^2u}{dz^2} + (\gamma_1 + \delta_1 z + \varepsilon_1 z^2)\frac{du}{dz} + (\alpha_1 z - q_1)u = 0$$

$$P_3(z) = p_3 \cdot (z - z_1)(z - z_2)(z - z_3) \quad z \to s_1 z + s_0 \quad P_3(z) = 1 \cdot z(z-1)(z-a)$$

1. General Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{du}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}u = 0$$

$$\begin{pmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & z \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta & \end{pmatrix}$$

2. Confluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right)\frac{du}{dz} + \frac{\alpha z - q}{z(z-1)}u = 0$$

The five Heun equations_2

3. Double-Confluent Heun equation

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z^2}+\frac{\delta}{z}+\varepsilon\right)\frac{du}{dz}+\frac{\alpha z-q}{z^2}u=0$$

4. Bi-Confluent Heun equation

$$\frac{d^2u}{dz^2}+\left(\frac{\gamma}{z}+\delta+\varepsilon z\right)\frac{du}{dz}+\frac{\alpha z-q}{z}u=0$$

5. Tri-Confluent Heun equation

$$\frac{d^2u}{dz^2}+\left(\gamma+\delta z+\varepsilon z^2\right)\frac{du}{dz}+(\alpha z-q)u=0$$

A fundamental solution of the auxiliary equation in terms of the general Heun equation

$$w = sn(\mu,k)^{2\sigma}\, dn(\nu,k_1)^{2\sigma} \times$$

$$\mathrm{HeunG}\left(1/k, q; \alpha, \beta, \gamma, \delta; sn(\mu,k)^2\right) \mathrm{HeunG}\left(k, q_1; \alpha, \beta, \gamma, \delta; dn^2(\nu,k_1)\right)$$

- **Bipolar limit of Cap-Cyclide coordinates for the Grad–Shafranov equation**

Bipolar limit of cap-cyclide coordinates for the Grad–Shafranov equation

$$A=-3/4 \qquad B=-2+16p^2/3$$

Radial part of the general solution

$$U=C_1U_1+C_2U_2$$

$$U_{1,2}=z^{3/4}\left(c_1u_1+c_2u_2\right)$$

In the limit $k=1$

$$z=\tanh^2(\mu)$$

$$u_1=\mathrm{HeunG}\left(1,\frac{3}{4}+p^2;\frac{1}{2},\frac{3}{2},2,\frac{1}{2};z\right)$$

$$u_2=\mathrm{HeunG}\left(0,-p^2;\frac{1}{2},\frac{3}{2},\frac{1}{2},2;1-z\right)$$

In terms of the ordinary hypergeometric functions

$$u_1 = \mathrm{HeunG}\left(1, \frac{3}{4} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z\right) = (1-z)^p \; {}_2F_1\left(\frac{1}{2} + p, \frac{3}{2} + p; 2; z\right)$$

Hypergeometric function in terms of the Legendre *P* function

$${}_2F_1(b, b+1; 2; y) = \frac{i(1-y)^{-b}}{b(b-1)\sqrt{y}} P_{b-1}^{1}\left(\frac{1+y}{1-y}\right), \quad 0 < y < 1$$

Fundamental solution

$$U_1 = \frac{2i\sqrt{2}}{4p^2 - 1} \sqrt{\sinh(2\mu)} \, P_{p-1/2}^{1}\left(\cosh(2\mu)\right)$$

Second fundamental solution

$$U_2 = \operatorname{sech}^{2p}(\mu) z^{3/4} \left(c_1 \cdot {}_2F_1\left(p+\frac{1}{2}, p+\frac{3}{2}; 2; z\right) + \right.$$

$$\left. + c_2 \cdot {}_2F_1\left(p+\frac{1}{2}, p+\frac{3}{2}; 2p+1; 1-z\right)\right)$$

Expression in terms of the Legendre Q function

$$c_1\, {}_2F_1\left(p+\frac{1}{2}, p+\frac{3}{2}; 2; y\right) + c_2\, {}_2F_1\left(p+\frac{1}{2}, p+\frac{3}{2}; 2p+1; 1-y\right) =$$

$$= \frac{i(1-y)^{-p-\frac{1}{2}} Q^1_{p-\frac{1}{2}}\left(\frac{y+1}{1-y}\right)}{\left(p-\frac{1}{2}\right)\left(p+\frac{1}{2}\right)\sqrt{y}}$$

$$2\mu = r$$

Second fundamental solution is rewritten as

$$U_2 = \frac{2i\sqrt{2}}{4p^2-1}\sqrt{\sinh(2\mu)}\, Q^1_{p-1/2}\left(\cosh(2\mu)\right)$$

Angular Solution

The general solution

$$V = C_3 V_1 + C_4 V_2 \qquad V_{1,2} = z^{3/4}\left(c_1 v_1 + c_2 v_2\right)$$

The only difference:

$$z = dn\left(\nu, k_1\right)^2$$

Elementary solution:

$$V = C_1 \sin(2\nu p) + C_2 \cos(2\nu p) \qquad 2\mu = r$$

Independent fundamental solutions

$$v_1 = c_1 \mathrm{HeunG}\left(k, \frac{3+9k}{16} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z \right) - i \cdot \ v_2$$

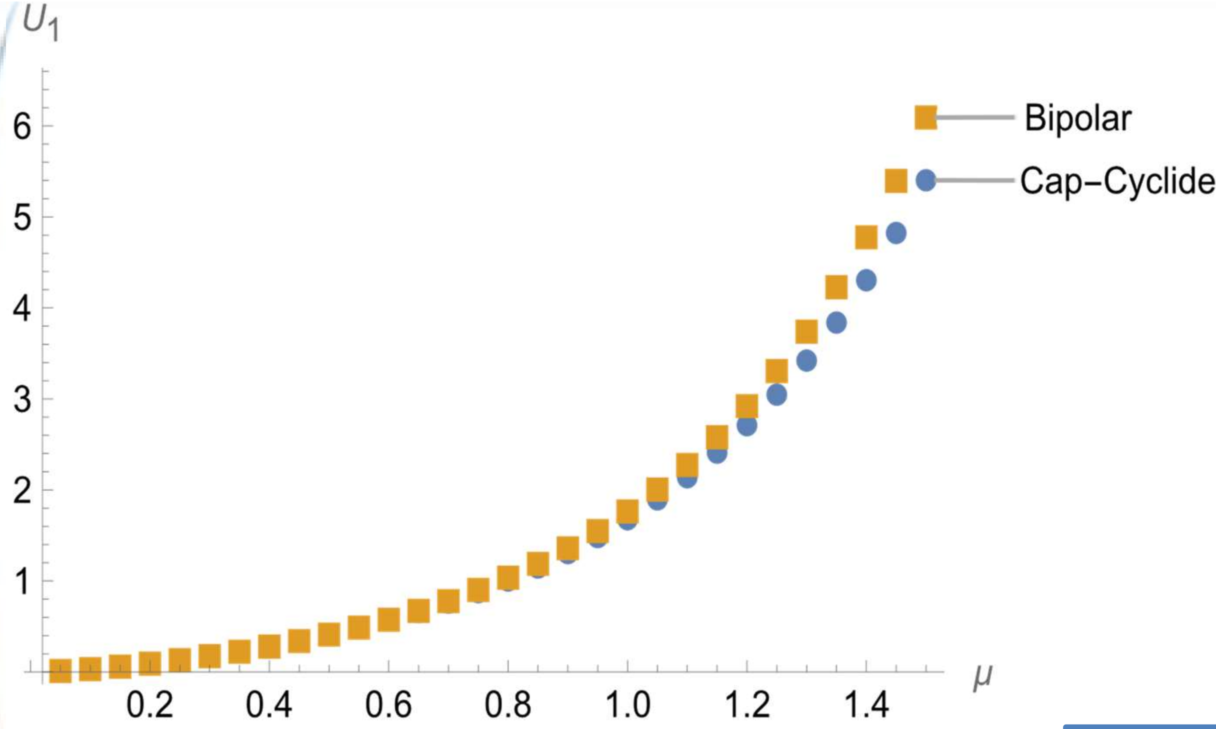
$$v_2 = \mathrm{HeunG}\left(1-k, \frac{9(1-k)}{16} - p^2; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 2; 1-z \right)$$

$$c_1 = \frac{z_0^{-3/4} + i \cdot \mathrm{HeunG}\left(1-k, \frac{9(1-k)}{16} - p^2; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 2; 1-z_0 \right)}{\mathrm{HeunG}\left(k, \frac{3+9k}{16} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z_0 \right)}$$

$$z_0 = dn\left(\frac{\pi}{4p}, 1-k \right)^2$$

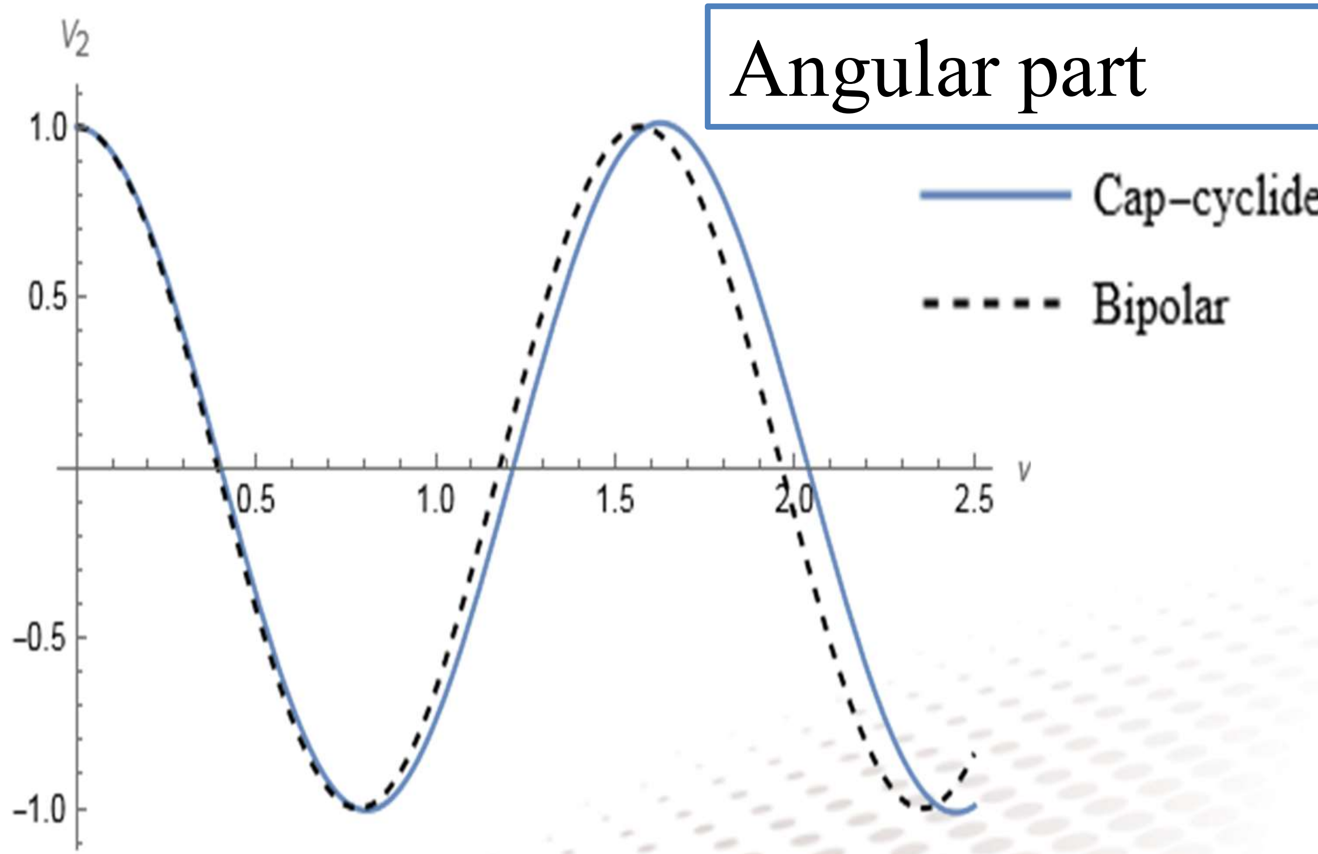
$$v_2\big|_{k\to 1} = \cos(2\nu p)$$

$$v_1\big|_{k\to 1} = \sin(2\nu p)$$



Radial part

Angular part



- Discussion
- ……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