

Grad–Shafranov equation in cap-cyclide coordinates: general Heun function Solution

Joint work with Prof. Flavio Crisanti, Italy

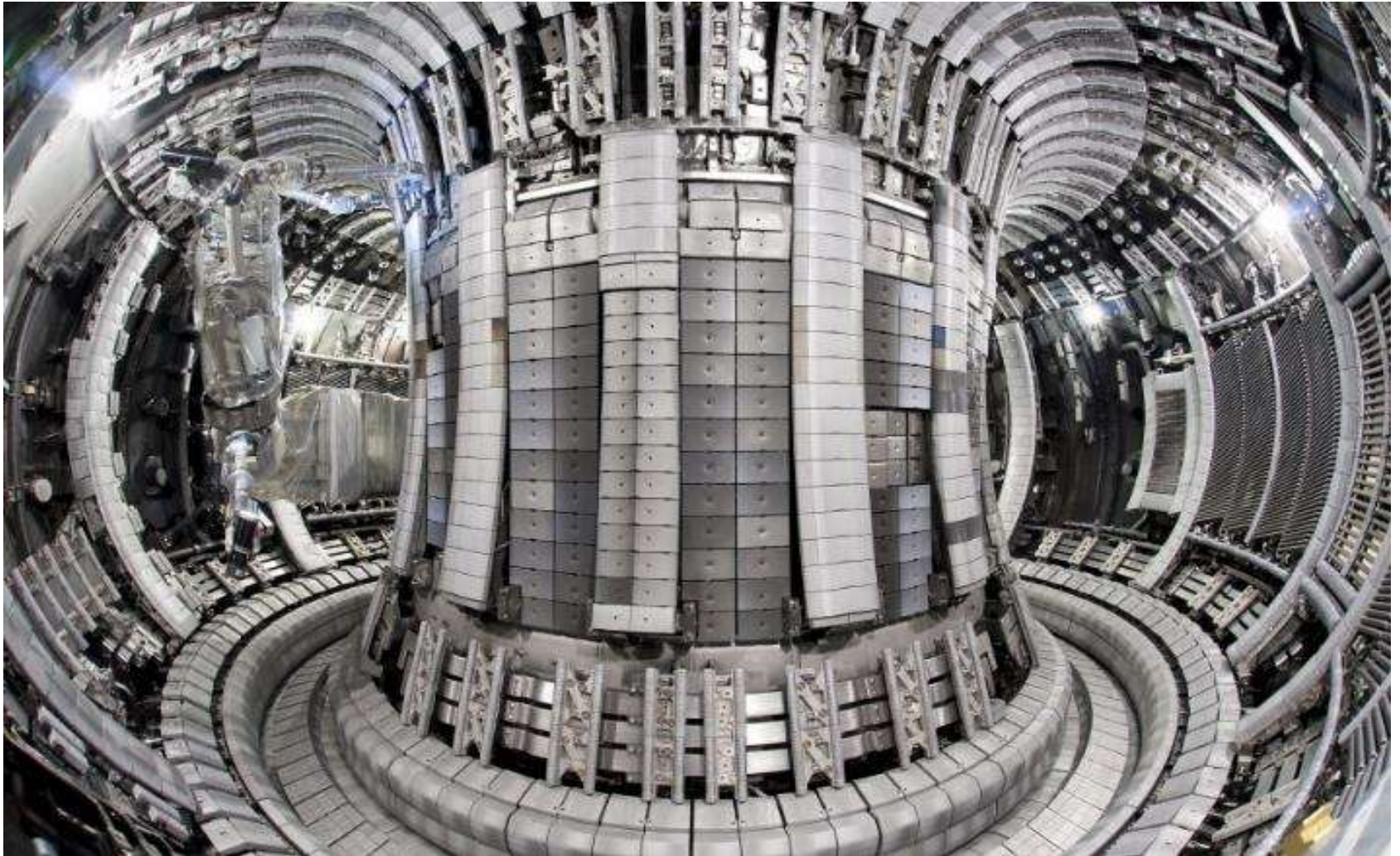
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Outline

- Tokamak (thermonuclear reactor) plasma shaping
- Cap-Cyclide coordinates
- Grad–Shafranov equation
Generalized Laplace equation
- Solution in Bipolar coordinates
- Solution in Cap-Cyclide coordinates
General Heun equation
- Bipolar limit of Cap-Cyclide coordinates for GSE
- Discussion

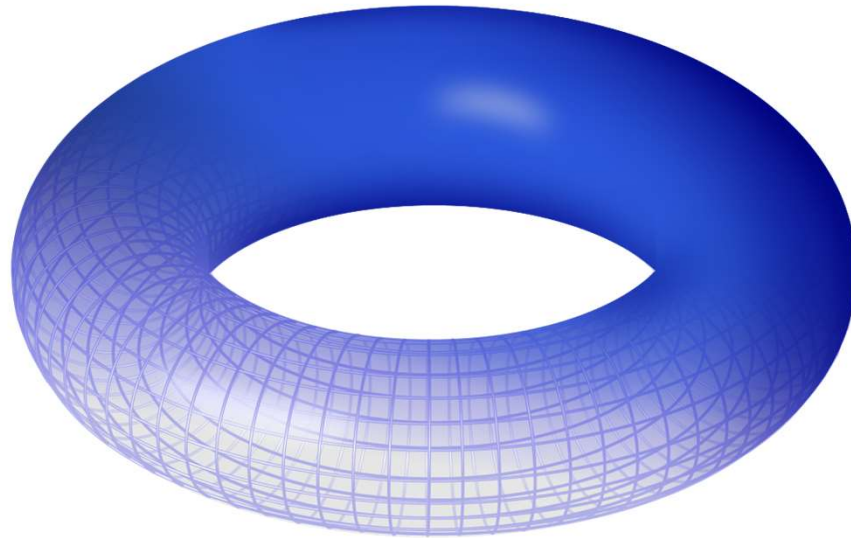
Plasma - Tokamak



A Tokamak is a device which uses a magnetic field to confine plasma in the shape of a torus

It is designed to produce controlled fusion energy. It is regarded as the main candidate for the role of a practical thermonuclear reactor

Initially: Torus configuration



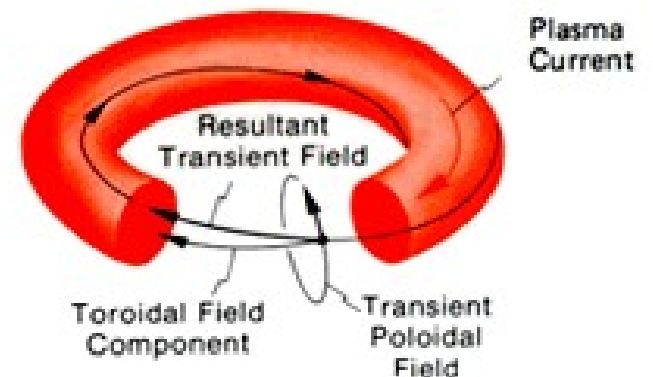
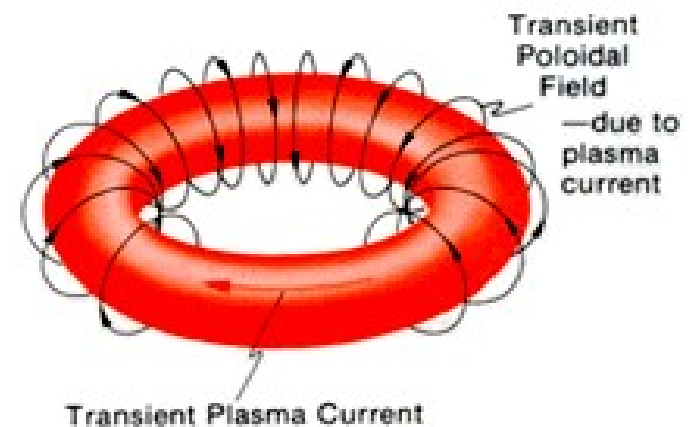
A stable plasma equilibrium requires magnetic field lines that spiral around the torus.

Lawson Criterion – triple product

Density x Temperature x Confinement time:

$$nT\tau$$

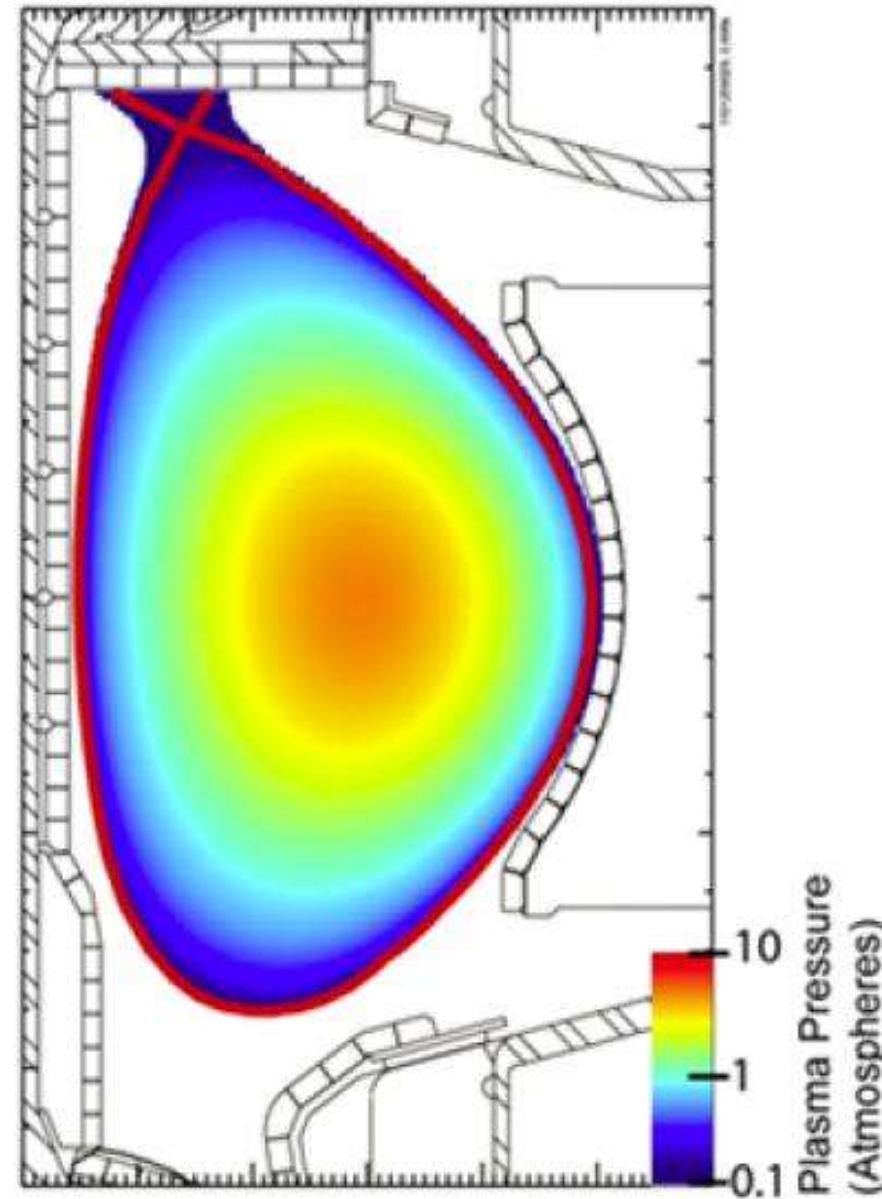
Relatively Constant Electric Current



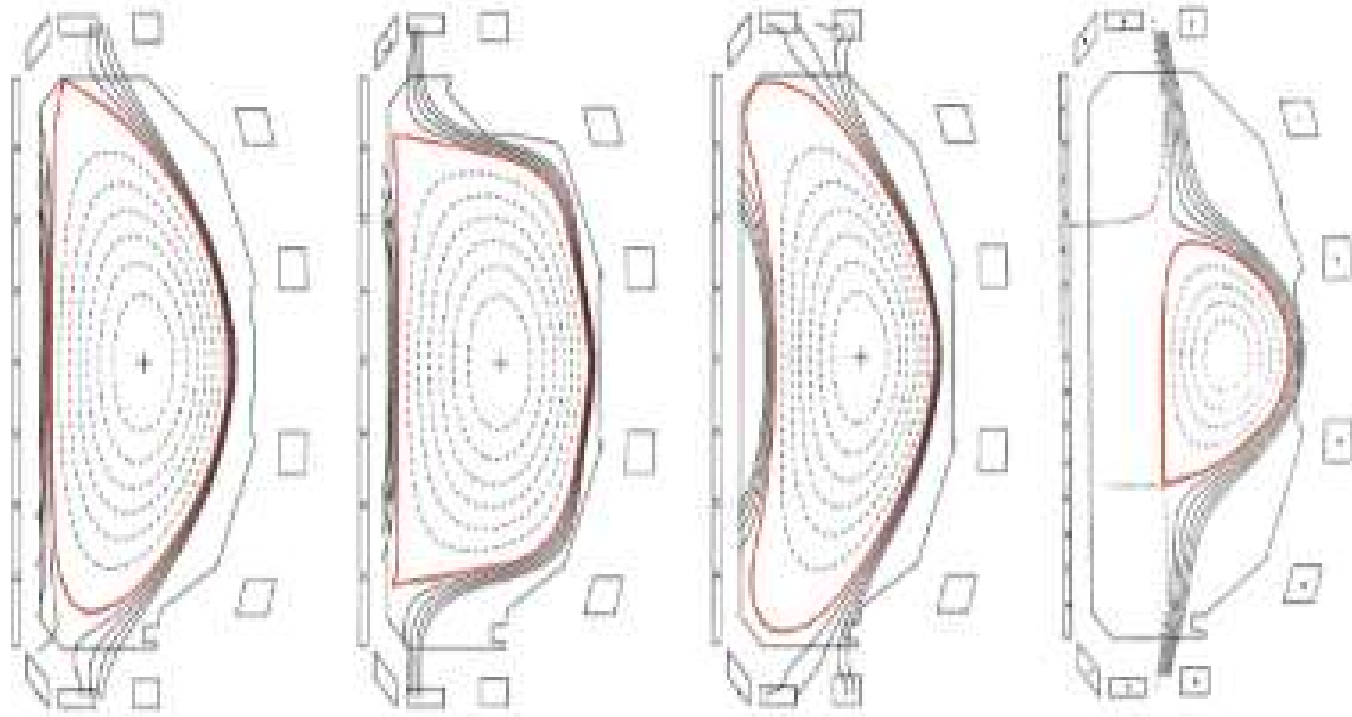
Plasma shaping - Geometric factors

Geometric factors influencing energy confinement time:

- Aspect ratio, $\Lambda = R / a$
- Plasma elongation, $k = b/a$,
where b is the height of the plasma measured from the equatorial plane
- Plasma triangularity, δ , the horizontal distance between the major radius and the x -point



Plasma Shaping



DIII-D

DIII-D

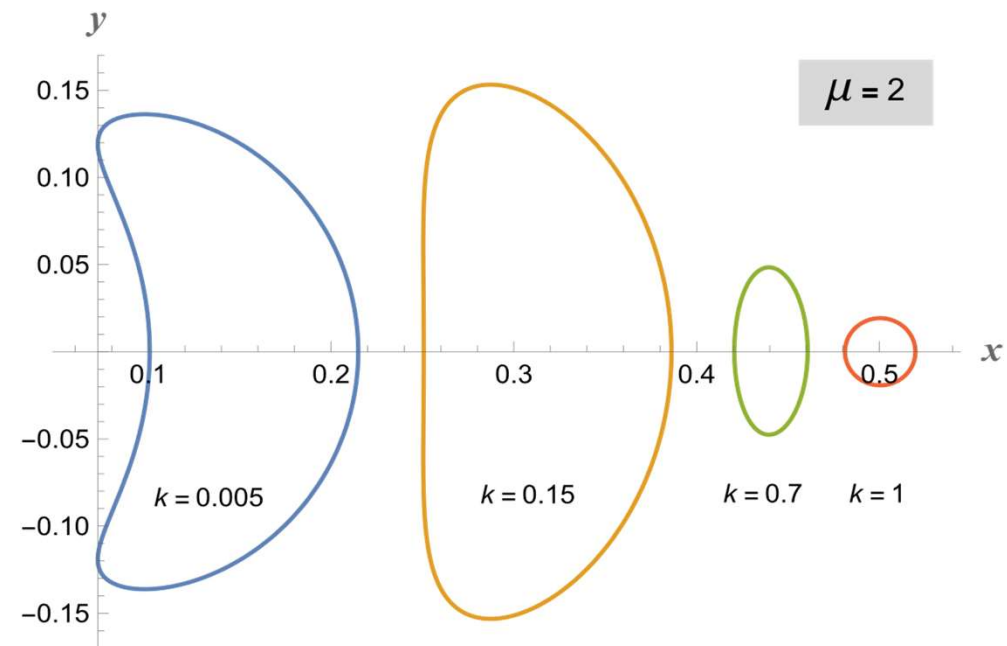
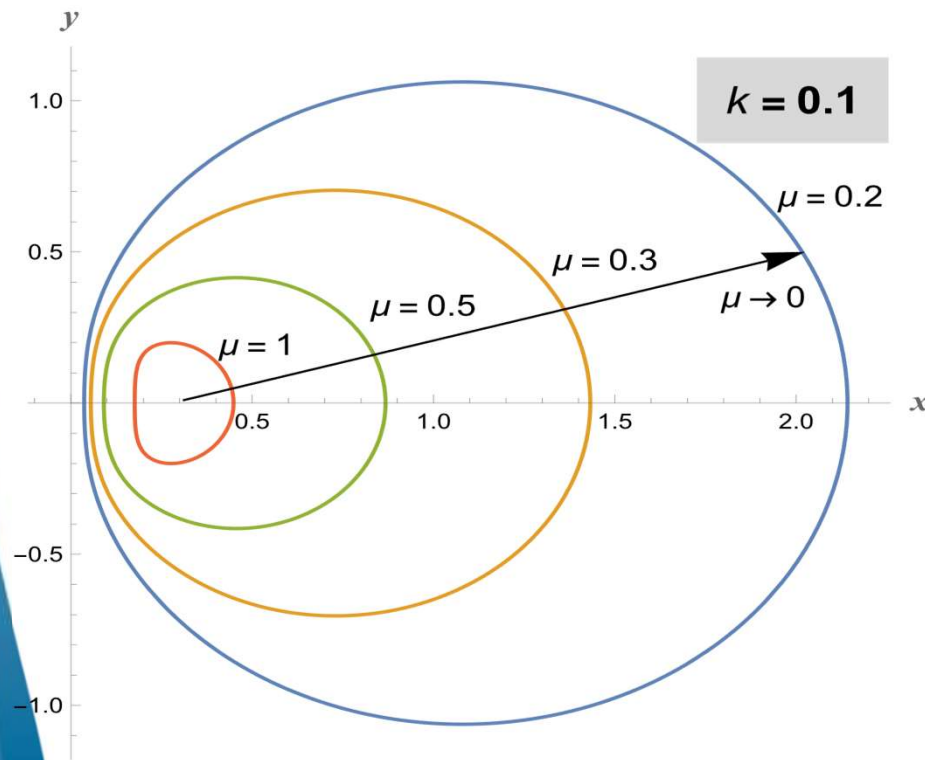
- Premier plasma shaping tokamak
- Modeled confinement time in excess of 300s
- Internal transport barrier optimization

The background of the slide is an abstract composition of overlapping, semi-transparent blue polygons. The colors range from a light, pale blue to a deeper, more saturated blue. The shapes are angular and geometric, creating a sense of depth and movement. The overall effect is a modern, clean, and professional aesthetic.

➤ **Cap-cyclide coordinates**

A natural coordinate system

Cap-cyclide coordinates



Cap-cyclide coordinates

*Real space coordinates (x, y, z)
are transformed to the coordinates (μ, ν, φ)*

$$x = \frac{\Lambda}{a\Gamma} \operatorname{sn}(\mu, k) \operatorname{dn}(\nu, k_1) \cos \varphi$$

$$y = \frac{\Lambda}{a\Gamma} \operatorname{sn}(\mu, k) \operatorname{dn}(\nu, k_1) \sin \varphi$$

$$z = \frac{k^{1/4} \Pi}{2a\Gamma} \quad k + k_1 = 1$$

Involved functions

$$\Lambda = 1 - dn^2(\mu, k) sn^2(\nu, k_1)$$

$$\Gamma = sn^2(\mu, k) dn^2(\nu, k_1) + \left(\Lambda / k^{1/4} + cn(\mu, k) dn(\mu, k) sn(\nu, k_1) cn(\nu, k_1) \right)^2$$

$$\Pi = \left(\Lambda^2 / k^{1/2} \right) - \left(sn^2(\mu, k) dn^2(\nu, k_1) + cn^2(\mu, k) dn^2(\mu, k) sn^2(\nu, k_1) cn^2(\nu, k_1) \right)$$

k is the parameter of elliptical integrals

$$k + k_1 = 1$$

k_1 is the complementary parameter of elliptical integrals

Complex function representation

Coordinate transformation: $(R, Z) \rightarrow (\mu, \nu)$

$$\frac{k^{1/4}}{2ai} \cdot \frac{1 + ik^{1/4} \operatorname{sn}(w)}{1 - ik^{1/4} \operatorname{sn}(w)} = R(\mu, \nu) + iZ(\mu, \nu)$$

$$w(\mu, \nu) = \mu + i\nu$$

Cauchy–Riemann conditions

$$\frac{\partial R}{\partial \mu} = \frac{\partial Z}{\partial \nu} \qquad \frac{\partial R}{\partial \nu} = -\frac{\partial Z}{\partial \mu}$$

- 
- The background of the slide features a complex, abstract geometric pattern of overlapping, semi-transparent blue polygons. The colors range from light sky blue to a deeper cerulean. The shapes are angular and layered, creating a sense of depth and movement. The overall effect is a modern, clean, and mathematical aesthetic.
- **Grad–Shafranov equation**
Generalized Laplace equation

Plasma equilibrium equation

$$\nabla P = \mathbf{J} \times \mathbf{B}$$

P is the kinetic plasma pressure

\mathbf{J} is the plasma current density

\mathbf{B} is the magnetic field

Flux function over a poloidal surface

$$\psi = \frac{1}{2\pi} \iint_{S_{\text{pol}}} \mathbf{B} \cdot d\mathbf{s}$$

Assuming axial symmetry and using cylindrical coordinates R, Z, φ

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = \mu_0 R J_\varphi$$

J_φ is the axisymmetric current density

This is the **Grad–Shafranov** equation

Grad-Shafranov and Laplace equations

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = 0$$

Transformation $\psi = R^{-\sigma_0/2} w$

Auxiliary equation $\frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{\partial Z^2} + \frac{A}{R^2} = 0$

Axisymmetric Laplace
equation

$$\sigma_0 = +1$$
$$A = 1/4$$

Grad-Shafranov
equation

$$\sigma_0 = -1, 3$$
$$A = -3/4$$

Generalized Laplace equation

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{\sigma_0}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = 0$$

Laplacian

$$\Delta w(R, Z) = \frac{\partial^2 w}{\partial R^2} + \frac{\partial^2 w}{\partial Z^2}$$

$$R = \varphi_1(q_1, q_2) \quad Z = \varphi_2(q_1, q_2)$$

$$\Delta w(q_1, q_2) = \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial q_1} \left(\frac{H_2}{H_1} \frac{\partial w}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{H_1}{H_2} \frac{\partial w}{\partial q_2} \right) \right]$$

Scale factors (Lamé coefficients)

$$H_i = \sqrt{\left(\frac{\partial \varphi_1}{\partial q_1} \right)^2 + \left(\frac{\partial \varphi_2}{\partial q_2} \right)^2} \quad i = 1, 2$$

The background features a complex, abstract geometric pattern of overlapping, semi-transparent blue polygons. The colors range from a light, pale blue to a deeper, more saturated blue. The shapes are irregular and angular, creating a sense of depth and movement. The overall effect is a modern, minimalist aesthetic.

➤ **Solution in Bipolar coordinates**

2D *bipolar*
coordinates r, θ

$$(x, y) = \left(\frac{a \sinh r}{\cosh r - \cos \theta}, \frac{a \sin \theta}{\cosh r - \cos \theta} \right)$$

2D Cartesian Laplacian

$$\Delta_C w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$



$$(R, Z) = (x, y): \Delta_B w = \frac{(\cosh r - \cos \theta)^2}{a^2} \left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} \right)$$

GSE is reduced to the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{A w}{\sinh^2 r} = 0$$

Separation of variables

$$w = u(r) \Theta(\theta)$$

Separation constant p

$$\Theta = \sin(p\theta + \theta_0)$$

Equation for the radial component

$$\frac{d^2 u}{dr^2} + \left(\frac{A}{\sinh^2 r} - p^2 \right) u = 0$$

General solution

$$u(r) = \sqrt{\sinh r} \left(C_1 P_{p-1/2}^q (\cosh r) + C_2 Q_{p-1/2}^q (\cosh r) \right)$$


$$q = \sqrt{\frac{1}{4} - A}$$

For axisymmetric Laplace equation

$$q = 0$$

For Grad–Shafranov equation

$$q = 1$$

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➤ **Solution in Cap-Cyclide coordinates**

Laplacian in cap-cyclide coordinates

$$\Delta_{CC} w = \frac{a^2 \Gamma^2}{\Lambda^2 \Omega^2} \left(\frac{\partial^2 w}{\partial \mu^2} + \frac{\partial^2 w}{\partial \nu^2} \right)$$

$$\Omega = \sqrt{\left(1 - \operatorname{sn}(\mu, k)^2\right) \operatorname{dn}(\nu, k_1)^2} \left(\operatorname{dn}(\nu, k_1)^2 - k \operatorname{sn}(\mu, k)^2 \right)$$

Auxiliary equation is rewritten as

$$\begin{aligned} \frac{\partial^2 w}{\partial \mu^2} + \frac{\partial^2 w}{\partial \nu^2} + A \left(\frac{1}{\operatorname{sn}(\mu, k)^2} + k \operatorname{sn}(\mu, k)^2 \right) - \\ - A \left(\frac{k}{\operatorname{dn}(\nu, k_1)^2} + \operatorname{dn}(\nu, k_1)^2 \right) = 0 \end{aligned}$$

Separation of variables

$$w(\mu, \nu) = U(\mu)V(\nu)$$

$$U''(\mu) + A \left(\frac{1}{\operatorname{sn}(\mu, k)^2} + k \operatorname{sn}(\mu, k)^2 + B \right) U(\mu) = 0$$

$$V''(\nu) - A \left(\frac{k}{\operatorname{dn}(\nu, k_1)^2} + \operatorname{dn}(\nu, k_1)^2 + B \right) V(\nu) = 0$$

Variable change

$$U = z^\sigma y(z) \quad z = \operatorname{sn}(\mu, k)^2$$

$$V = z^\sigma y(z) \quad z = \operatorname{dn}(\nu, k_1)^2$$

$$\sigma = \frac{1}{4} \left(1 \pm \sqrt{1 - 4A} \right) = \frac{\sigma_0}{4}$$

General Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0$$

$$(\gamma, \delta, \varepsilon, \alpha, \beta) = \left(\frac{1+4\sigma}{2}, \frac{1}{2}, \frac{1}{2}, 2\sigma, \frac{1}{2} \right)$$

$$\text{for } U: \quad a = \frac{1}{k}, \quad q = \frac{2(1+k)\sigma - A(B+k+1)}{4k}$$

$$\text{for } V: \quad a = k, \quad q = \frac{2(1+k)\sigma - A(B+k+1)}{4}$$

The five Heun equations_1

- Equations of the Heun class

$$(p_0 + p_1z + p_2z^2 + p_3z^3) \frac{d^2u}{dz^2} + (\gamma_1 + \delta_1z + \varepsilon_1z^2) \frac{du}{dz} + (\alpha_1z - q_1)u = 0$$

$$P_3(z) = p_3 \cdot (z - z_1)(z - z_2)(z - z_3) \quad z \rightarrow s_1z + s_0 \quad P_3(z) = 1 \cdot z(z - 1)(z - a)$$

1. General Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a} \right) \frac{du}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} u = 0$$

$$\begin{pmatrix} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha z \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta \end{pmatrix}$$

2. Confluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon \right) \frac{du}{dz} + \frac{\alpha z - q}{z(z-1)} u = 0$$

The five Heun equations_2

3. Double-Confluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z^2} + \frac{\delta}{z} + \varepsilon \right) \frac{du}{dz} + \frac{\alpha z - q}{z^2} u = 0$$

4. Bi-Confluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\frac{\gamma}{z} + \delta + \varepsilon z \right) \frac{du}{dz} + \frac{\alpha z - q}{z} u = 0$$

5. Tri-Confluent Heun equation

$$\frac{d^2u}{dz^2} + \left(\gamma + \delta z + \varepsilon z^2 \right) \frac{du}{dz} + (\alpha z - q)u = 0$$

**A fundamental solution of the auxiliary equation
in terms of the general Heun equation**

$$w = sn(\mu, k)^{2\sigma} dn(\nu, k_1)^{2\sigma} \times$$

$$\text{HeunG}\left(1/k, q; \alpha, \beta, \gamma, \delta; sn(\mu, k)^2\right) \text{HeunG}\left(k, q_1; \alpha, \beta, \gamma, \delta; dn^2(\nu, k_1)\right)$$

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➤ **Bipolar limit of Cap-Cyclide coordinates
for the Grad–Shafranov equation**

Bipolar limit of cap-cyclide coordinates for the Grad–Shafranov equation

$$A = -3/4 \qquad B = -2 + 16p^2/3$$

Radial part of the general solution

$$U = C_1 U_1 + C_2 U_2$$

$$U_{1,2} = z^{3/4} (c_1 u_1 + c_2 u_2)$$

In the limit $k = 1$ $z = \tanh^2(\mu)$

$$u_1 = \text{HeunG} \left(1, \frac{3}{4} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z \right)$$

$$u_2 = \text{HeunG} \left(0, -p^2; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 2; 1-z \right)$$

In terms of the ordinary hypergeometric functions

$$u_1 = \text{HeunG}\left(1, \frac{3}{4} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z\right) = (1-z)^p {}_2F_1\left(\frac{1}{2} + p, \frac{3}{2} + p; 2; z\right)$$

Hypergeometric function in terms of the Legendre
 P function

$${}_2F_1(b, b+1; 2; y) = \frac{i(1-y)^{-b}}{b(b-1)\sqrt{y}} P_{b-1}^1\left(\frac{1+y}{1-y}\right), \quad 0 < y < 1$$

Fundamental solution

$$U_1 = \frac{2i\sqrt{2}}{4p^2 - 1} \sqrt{\sinh(2\mu)} P_{p-1/2}^1(\cosh(2\mu))$$

Second fundamental solution

$$U_2 = \operatorname{sech}^{2p}(\mu) z^{3/4} \left(c_1 {}_2F_1\left(p + \frac{1}{2}, p + \frac{3}{2}; 2; z\right) + c_2 {}_2F_1\left(p + \frac{1}{2}, p + \frac{3}{2}; 2p + 1; 1 - z\right) \right)$$

Expression in terms of the Legendre Q function

$$c_1 {}_2F_1\left(p + \frac{1}{2}, p + \frac{3}{2}; 2; y\right) + c_2 {}_2F_1\left(p + \frac{1}{2}, p + \frac{3}{2}; 2p + 1; 1 - y\right) = \frac{i(1-y)^{-p-\frac{1}{2}} Q_{p-\frac{1}{2}}^1\left(\frac{y+1}{1-y}\right)}{\left(p - \frac{1}{2}\right)\left(p + \frac{1}{2}\right)\sqrt{y}}$$

$2\mu = r$

Second fundamental solution is rewritten as

$$U_2 = \frac{2i\sqrt{2}}{4p^2 - 1} \sqrt{\sinh(2\mu)} Q_{p-1/2}^1(\cosh(2\mu))$$

Angular Solution

The general solution

$$V = C_3 V_1 + C_4 V_2 \quad V_{1,2} = z^{3/4} (c_1 v_1 + c_2 v_2)$$

The only difference:

$$z = dn(v, k_1)^2$$

Elementary solution:

$$V = C_1 \sin(2vp) + C_2 \cos(2vp) \quad 2\mu = r$$

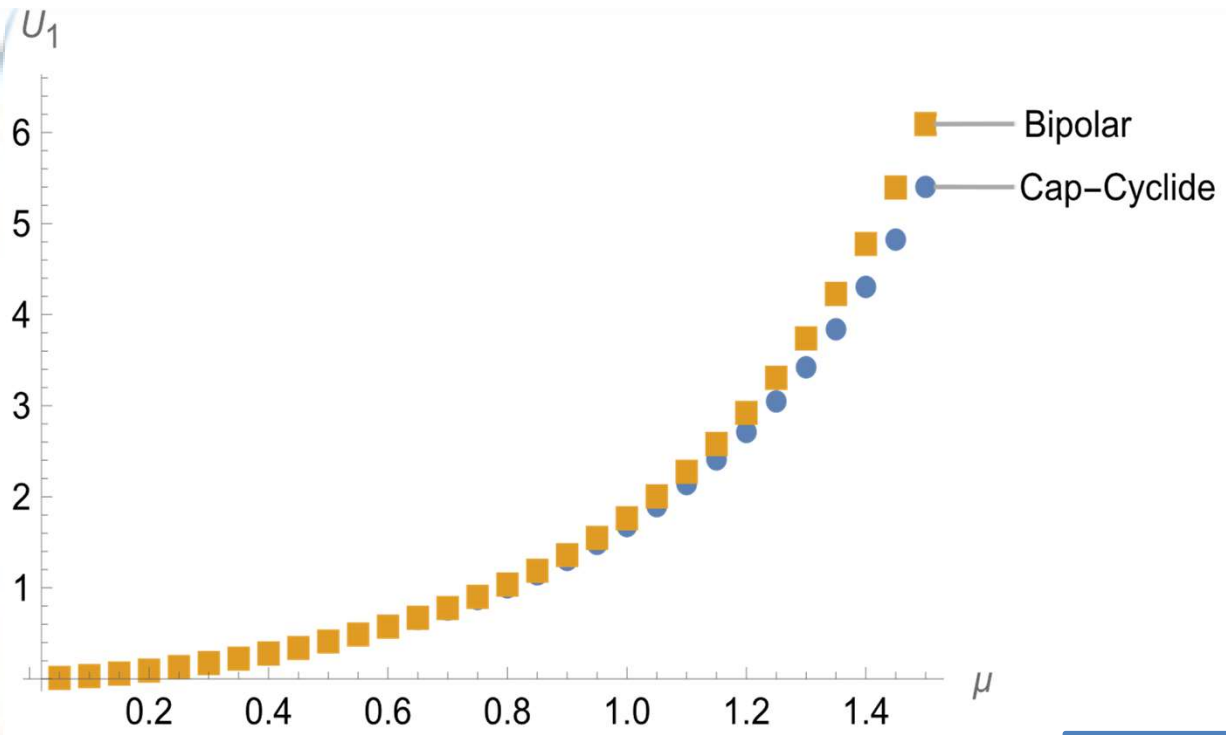
Independent fundamental solutions

$$v_1 = c_1 \text{HeunG} \left(k, \frac{3+9k}{16} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z \right) - i \cdot v_2$$

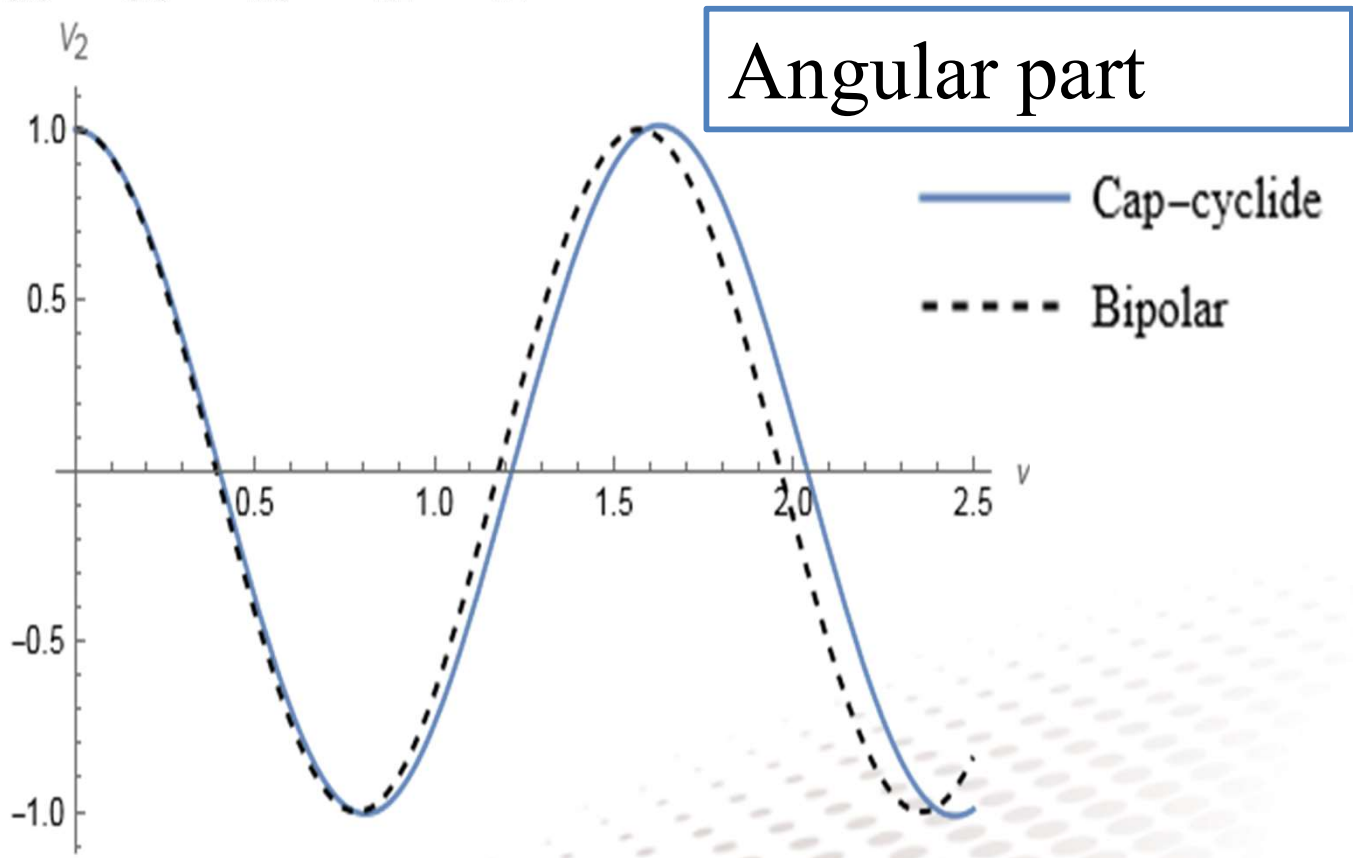
$$v_2 = \text{HeunG} \left(1-k, \frac{9(1-k)}{16} - p^2; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 2; 1-z \right)$$

$$c_1 = \frac{z_0^{-3/4} + i \cdot \text{HeunG} \left(1-k, \frac{9(1-k)}{16} - p^2; \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, 2; 1-z_0 \right)}{\text{HeunG} \left(k, \frac{3+9k}{16} + p^2; \frac{1}{2}, \frac{3}{2}, 2, \frac{1}{2}; z_0 \right)}$$

$$z_0 = dn \left(\frac{\pi}{4p}, 1-k \right)^2 \quad \begin{aligned} v_2|_{k \rightarrow 1} &= \cos(2\nu p) \\ v_1|_{k \rightarrow 1} &= \sin(2\nu p) \end{aligned}$$



Radial part



Angular part

The background of the slide is a light blue and white geometric pattern of overlapping polygons, creating a crystalline or faceted appearance. The colors transition from a pale blue on the right to a deeper blue on the left.

➤ Discussion

➤

A scenic landscape photograph featuring a dense forest of green trees in the foreground. In the middle ground, a city with several buildings is visible. The background is dominated by large, snow-capped mountains under a clear blue sky with a few wispy clouds.

Thank you

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