Maxwell-Jüttner distribution for rotating spinning particle gas

Nikita N. Levin

Tomsk State University

20/08/2023

Presentation Overview

1. Introduction
   - Problem & motivation
   - Historical notes
2. Microscopic description
3. Statistical mechanics
   - One-particle distribution function
   - Angular distribution function
4. Thermodynamics
5. Conclusion

Problem & motivation

Problem

We consider the statistical mechanics and thermodynamics of rotating with constant angular velocity ideal gas of relativistic classical particles with nonzero spin at finite temperature.

Motivation

- Development of quasi-classical theory;
- Chiral effects in rotating systems;
- Universal description of particles of all spins;
- Explicit expressions of functions and quantities.

The talk is based on joint work with D.S. Kaparulin, arXiv:2302.05639.

Historical notes

- Maxwell-Boltzmann distribution (Maxwell, 1878):

$$f(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{z_0} \exp\left(- \frac{\boldsymbol{p}^2}{2mkT} + \frac{m[\boldsymbol{\Omega}, \boldsymbol{x}]^2}{2kT} \right); \quad (1)$$

- Generalized Maxwell-Boltzmann distribution (Bubenchikov, Kaparulin, Nosyrev, 2022): arXiv:2205.06682.

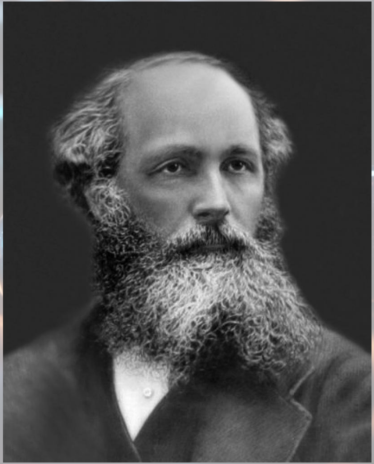

Figure: James Clerk Maxwell

Historical notes

- Gibbs distribution for rotating system (Gibbs, 1902):

$$f(\boldsymbol{x},\boldsymbol{p},....)=\frac{1}{z_0}\exp\left(-\frac{\varepsilon(\boldsymbol{x},\boldsymbol{p},....)-(\boldsymbol{\Omega},\boldsymbol{j}(\boldsymbol{x},\boldsymbol{p},....))}{kT}\right); \qquad (2)$$

- Maxwell-Jüttner distribution (Jüttner, 1911; Synge, 1957):

$$f(\boldsymbol{x},\boldsymbol{p})=\frac{1}{z_0}\exp\left(-\frac{mc^2\gamma}{kT\gamma}\right), \qquad \frac{1}{\gamma}=\sqrt{1-\frac{[\boldsymbol{\Omega},\boldsymbol{x}]^2}{c^2}}. \qquad (3)$$

- Quantum distributions: arXiv:0911.0864, 1812.08886, 1811.04409, etc.

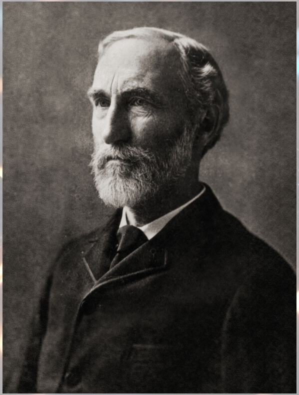

Figure: Josiah Willard Gibbs

Model of relativistic spinning particle

Microscopic description problems

- Construction of the phase-space and invariant measure on it.
- Expressing energy-momentum in terms of phase-space coordinates.

We consider a model of massive relativistic spinning particle with the mass m and spin s with the action functional (Deriglazov, Pupasov-Maksimov, 2013):

$$S = \int \Big\{ (p, \dot{x}) + (\pi, \dot{\omega}) - \lambda_p((p,p) + m^2c^2) - \lambda_\omega((\omega,\omega) - a^2) - \\ -\lambda_\pi((\pi,\pi) - b^2) - \lambda_{p\omega}(p,\omega) - \lambda_{p\pi}(p,\pi) - \lambda_{\omega\pi}(\omega,\pi) \Big\} d\tau \,, \tag{4}$$

with x^μ being space-time position of the particle, p^μ - canonically conjugated momenta, ω^μ and π^μ - spinning **sector** variables, $\mu = 0, 1, 2, 3$; $\lambda_p, \ldots, \lambda_{\omega\pi}$ - Lagrange multipliers, τ - time.

Parameters a and b determine the particle's spin by the rule: $a^2 b^2 = \hbar^2 \Sigma^2 = \hbar^2 s(s+1)$.

Model of relativistic spinning particle

Since physical observables of the model are functions of momentum and total angular momentum, we introduce:

$$j^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu} + s^{\mu\nu}, \qquad s^{\mu\nu} = \omega^{\mu}\pi^{\nu} - \omega^{\nu}\pi^{\mu}\,, \tag{5}$$

where $j^{\mu\nu}$ - total angular momentum, $s^{\mu\nu}$ - spin angular momentum; $\mu\,,\nu = 0,1,2,3$

The physical phase-space of the considered model is a $\mathbb{R}^6 \times \mathbb{S}^2$ bundle with three spacial coordinates of the particle, three linear momentum components and two variables, that determine the direction of spin.

Parametrization of physical states

By using (1+3) decomposition, we express space-time quantities in phase-space coordinate terms:

$$x^\mu = (ct, \boldsymbol{x})\,, \quad p^\mu = mc\gamma(1, \boldsymbol{\beta})\,, \quad s^{\mu\nu} = \hbar\Sigma(\gamma[\boldsymbol{\beta}, \boldsymbol{c}], \gamma\boldsymbol{c} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}, \boldsymbol{c}))\,. \tag{6}$$

where **bold symbols** denote spacial coordinates of the corresponding vectors, γ and β being relativistic factors, c - speed of light, and $\boldsymbol{c} = \boldsymbol{c}(\alpha^\circ, \beta^\circ)$ - Euclidean unit vector:

$$\boldsymbol{c}(\alpha^\circ, \beta^\circ) = (\cos\alpha^\circ \sin\beta^\circ, \sin\alpha^\circ \sin\beta^\circ, \cos\beta^\circ)\,. \tag{7}$$

In the result, the coordinates on the physical phase-space: $\boldsymbol{x}$, $\boldsymbol{p}$, and α°, β°.
And invariant measure on it:

$$\mathrm{d}\Gamma = \frac{(2s+1)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}\sin\beta^\circ \mathrm{d}\alpha^\circ \mathrm{d}\beta^\circ}{4\pi(2\pi\hbar)^3}\,. \tag{8}$$

One-particle distribution function

For the construction of the statistical mechanics, we use the special form of Gibbs distribution for a rotating (with constant angular velocity $\boldsymbol{\Omega}$ and temperature T) system, which mass center is at rest:

$$f_0(\varepsilon, \boldsymbol{j}) = \frac{1}{z_0} \exp\left(- \frac{\varepsilon - (\boldsymbol{\Omega}, \boldsymbol{j})}{kT} \right), \tag{9}$$

with $\varepsilon = mc^2\gamma$ - particle energy, k - Boltzmann constant, z_0 - partition function. Zero index symbolizes one-particle distribution.

By systematic substitution of quantities (5), (6) in (9), we obtain the generalized Maxwell-Jüttner distribution for a rotating gas of spinning particles:

$$f_0(\boldsymbol{x},\, \boldsymbol{p},\, \alpha^\circ,\, \beta^\circ) = \frac{1}{z_0} \exp\left(- \frac{mc^2\gamma}{kT} + \frac{mc\gamma(\boldsymbol{\Omega}, \boldsymbol{x}, \boldsymbol{\beta})}{kT} + \frac{\hbar\Sigma}{kT}(\boldsymbol{\Omega}, \gamma\boldsymbol{c} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}, \boldsymbol{c})) \right). \tag{10}$$

One-particle distribution function

The partition function (per particle) z_0 is determined from the normalizing condition with respect to invariant measure:

$$z_0 = \int \exp\left(-\frac{mc^2\gamma}{kT} + \frac{mc\gamma(\boldsymbol{\Omega}, \boldsymbol{x}, \boldsymbol{\beta})}{kT} + \frac{\hbar\Sigma}{kT}(\boldsymbol{\Omega}, \gamma\boldsymbol{c} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}, \boldsymbol{c}))\right) \mathrm{d}\Gamma. \tag{11}$$

Immediate conclusions:

- The distribution function by spins is asymmetric if Ω is nonzero;
- Spinning corrections are proportional to ratio $\vartheta \equiv \hbar\Sigma\Omega/kT$.

From formula (10) special distributions can be derived, such as **position-momenta distribution** (arXiv:0911.0864) and **angular distribution**.

Distribution function in the rotating frame

Distribution function (10) can be rewritten in the rotating frame connected to the reservoir with gas. After Lorentz transformations applied, the distribution function taking the following form:

$$f_0 = \frac{1}{z_0} \exp\left(-\frac{mc^2\gamma}{kT\gamma} + \frac{\hbar\Sigma\gamma}{kT}(\boldsymbol{\Omega}, \gamma\boldsymbol{c} + \gamma[\boldsymbol{\beta}, [\boldsymbol{\beta}, \boldsymbol{c}]] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}, \boldsymbol{c}))\right), \tag{12}$$

where vectors $\boldsymbol{\beta}$, $\boldsymbol{c}$ and factor γ - determined in the rotation frame, and the notation is used:

$$\boldsymbol{\beta} = \frac{1}{c}[\boldsymbol{\Omega}, \boldsymbol{x}], \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{13}$$

The leading term of the exponent determines the usual Maxwell-Jüttner distribution for spinless particle, and the second contribution determines the spinning correction.

The translational and spinning degree of freedom do not decouple even in the rotating frame. This corresponds to a special form of spin-orbital interaction in relativistic gas.

Angular distribution function

Why are we interested in it?

Because the angular variables α° and β° determine the direction of particle's spin. In other words, the current distribution function characterizes the behavior of spinning degree of freedom in the rotating systems.

The angular distribution function can be explicitly derived from (12), by the following rule:

$$f_{\alpha\beta}(\alpha^\circ, \beta^\circ) = \frac{1}{z_{\alpha\beta}} \int \exp\left(-\frac{\gamma}{\theta\gamma} + \vartheta\gamma\Phi \right) \frac{\gamma \mathrm{d}\boldsymbol{p}}{m^3 c^3}\,, \tag{14}$$

where $\theta = kT/mc^2$ - dimensionless temperature, $z_{\alpha\beta}$ - partition function, $\Phi(\alpha^\circ, \beta^\circ, \xi^\circ, \zeta^\circ, \gamma)$ - term, proportional to spinning corrections (ξ°, ζ° - angles in auxiliary rotating frame, where $\boldsymbol{\Omega}$ is co-directed with rotating axis), $\mathrm{d}\boldsymbol{p}(\xi^\circ, \zeta^\circ, \gamma)$ - volume element in the momenta space.

We consider particle's position $\boldsymbol{x}$ here as a parameter.

Partition function

Normalizing condition for $f_{\alpha\beta}$:

$$\frac{1}{4\pi}\int f_{\alpha\beta}\sin\beta^\circ \mathrm{d}\alpha^\circ \mathrm{d}\beta^\circ = 1\,. \qquad (15)$$

Condition (15) leads us to the explicit form of partition function $z_{\alpha\beta}$:

$$z_{\alpha\beta} = 4\pi\gamma\left\{\theta\left(1+\vartheta^2\frac{1+10\theta^2\beta^2}{6}\right)\mathrm{K}_2(1/\theta)+\theta^2\vartheta^2\frac{1+\beta^2(1+30\theta^2)}{3}\mathrm{K}_3(1/\theta)\right\}, \qquad (16)$$

where $\mathrm{K}_a(1/\theta)$, $a=2,3$ - modified Bessel's functions of the second kind, $\theta=\gamma\theta$, $\vartheta=\gamma\vartheta$.

Generally speaking, obtained partition function (16) can not be called such yet, because we need to integrate it by particle's position.

Angular distribution functions by a singular variable

Combination of formulas (14) and (16) gives us a desired angular distribution function. Since this is a two-variable function, the plot of it is going to be three-dimentional, that is not so convenient for analysis. For this reason, we factorize it into two separate functions.

General form:

$$f_\alpha(\alpha^\circ) = \frac{1}{2}\int f_{\alpha\beta} \sin\beta^\circ \mathrm{d}\beta^\circ :$$

$$f_\beta(\beta^\circ) = \frac{1}{2\pi}\int f_{\alpha\beta} \mathrm{d}\alpha^\circ :$$

Leading order:

$$f_\alpha = 1 - \frac{5\theta K_2(1/\theta) + (1+30\theta^2)K_3(1/\theta)}{6\theta K_2(1/\theta)}\theta\beta^2\vartheta^2 \cos(2\alpha^\circ) + \dots \quad (17)$$

$$f_\beta = 1 + \frac{(1-2\theta)K_2(1/\theta) + 2K_3(1/\theta)}{3K_2(1/\theta)}\vartheta \cos\beta^\circ + \dots \quad (18)$$

Angular distribution plots

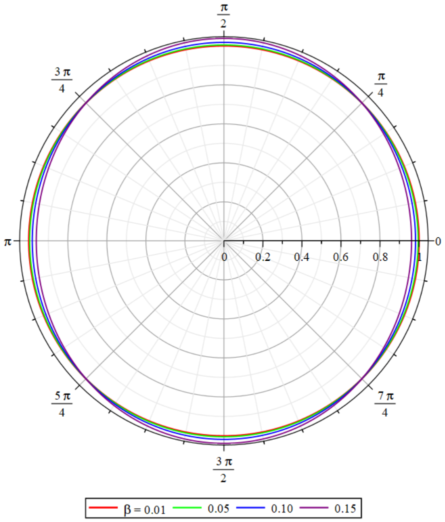

Figure: f_α distribution with $\theta = 1$, $\vartheta = 0.3$.

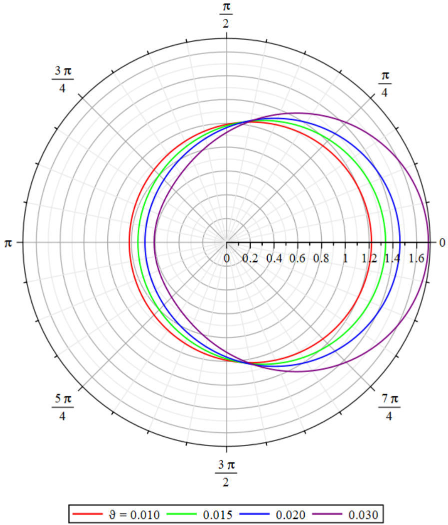

Figure: f_β distribution with $\beta = 0.001$, $\theta = 10$.

Thermodynamics of rotating gas

Thermodynamic problems

- Computation of the partition function z_0.
- Obtaining explicit expressions for thermodynamic quantities such as total angular momentum j, entropy S, and internal energy U.

We consider the thermodynamics of a slowly rotating relativistic ideal gas of spinning particles in a limited cylinder with the radius R and volume V. We assume that linear velocity of reservoir rotation is much smaller than the speed of light ($\beta_m = \Omega R/c \ll 1$) and magnitude of chiral effects is small ($\vartheta \ll 1$).

According to the expression (11), after integration, with respect to the reservoir's shape, we have:

$$z_0 = \frac{4\pi m^3 c^3 (2s+1) V}{(2\pi\hbar)^3} \theta \mathrm{K}_2(\theta) \left(1 + \beta_m^2 \frac{\mathrm{K}_3(\theta) - 2\theta \mathrm{K}_2(\theta)}{4\theta \mathrm{K}_2(\theta)} + \vartheta^2 \frac{\mathrm{K}_2(\theta) + 2\theta \mathrm{K}_3(\theta)}{6\mathrm{K}_2(\theta)} + \dots \right). \quad (19)$$

General thermodynamic formulas

To obtain early announced thermodynamic quantities, we have to determine partition function z of classical ideal gas (consisting of N particles) and thermodynamic potential Φ by the rules:

$$z = \frac{1}{N!}(z_0)^N \,, \qquad \Phi = -kT \ln z \,. \tag{20}$$

Formulas (19) and (20) lead us to the explicit form of thermodynamic potential:

$$\Phi = -kTN \left\{ \ln \frac{4\pi m^3 c^3 (2s+1) V}{N(2\pi\hbar)^3} \theta \mathrm{K}_2(\theta) + \beta_m^2 \frac{\mathrm{K}_3(\theta) - 2\theta \mathrm{K}_2(\theta)}{4\theta \mathrm{K}_2(\theta)} + \vartheta^2 \frac{\mathrm{K}_2(\theta) + 2\theta \mathrm{K}_3(\theta)}{6\mathrm{K}_2(\theta)} + 1 \right\} . \tag{21}$$

As in (19), the leading term of the expression determines non-rotating relativistic gas, and two others terms account rotational and spinning corrections.

Thermodynamic quantities

General form:

$$j = -\frac{\partial \Phi}{\partial \Omega}\ ;$$

$$S = -\frac{\partial \Phi}{\partial \theta}\ ;$$

Explicit form:

$$j = N\left\{\frac{m\Omega R^2}{2}\,\frac{\mathrm{K}_3(\theta) - 2\theta \mathrm{K}_2(\theta)}{\mathrm{K}_2(\theta)} + \frac{\hbar\Sigma\vartheta}{3}\,\frac{\mathrm{K}_2(\theta) + 2\theta \mathrm{K}_3(\theta)}{\mathrm{K}_2(\theta)}\right\}. \tag{22}$$

$$S = kN\left\{\ln\frac{8\pi m^3 c^3 (2s+1) V}{N(2\pi\hbar)^3}\theta^3 + \frac{1}{2}\beta_m^2 + \frac{4}{3}\theta^2\vartheta^2 + 4\right\}. \tag{23}$$

Both terms in the (22) have clear physical interpretation. The $mR^2\Omega/2$ contribution have a sense of orbital angular momentum of spinless relativistic gas. It is interesting to note that the momentum of inertia of the gas depends on temperature. The second term correspond to the average value of spin $\langle s \rangle$. Internal energy can be found from connection: $\Phi = U - TS - \Omega j$.

Entropy (23) was computed in the ultra-relativistic limit.

Average value of spin and spin polarization

In the low temperature limit, the average value of spin in (22) corresponds to the non-relativistic polarization expression (Bubenchikov, Kaparulin, Nosyrev, 2022):

$$\langle s \rangle = \frac{\hbar \Sigma \vartheta}{3} \frac{\mathrm{K}_2(\theta) + 2\theta \mathrm{K}_3(\theta)}{\mathrm{K}_2(\theta)} \xrightarrow{\theta \to 0} \frac{\hbar \Sigma \vartheta}{3} . \tag{24}$$

Also, we can consider the specific rotational polarizability χ of spinning degree of freedom, that determines by the rule and has an obvious form:

$$\chi \equiv \frac{\partial \langle s \rangle}{\partial \Omega}\bigg|_{\Omega=0} = \frac{\hbar^2 \Sigma^2}{3mc^2} \frac{\mathrm{K}_2(\theta) + 2\theta \mathrm{K}_3(\theta)}{\theta \mathrm{K}_2(\theta)} . \tag{25}$$

Spin polarization plot

With respect to constants, we can obtain the visual representation of $\chi(\theta)$ dependence.

The polarizability decreases for low temperatures and increases for high ones, with a visible minimum at the intermediate value of temperature: $\theta_m = 0.3859..$, with the extreme value: $\chi_m = 2.3044..\hbar^2\Sigma^2/mc^2$.

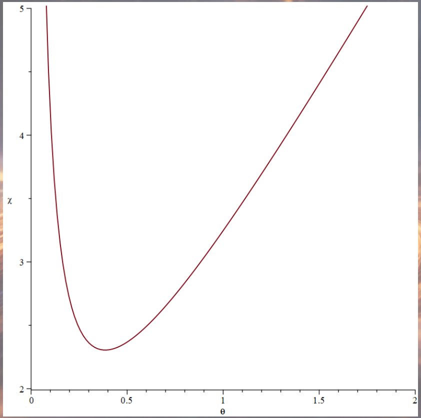

Figure: Specific polarizability of relativistic gas

Conclusion and outlook

The following results were obtained:

- Statistical mechanics and thermodynamics of classical rotating ideal gas of relativistic spinning particles have been developed;
- A Generalized Maxwell-Jüttner distribution function by positions, momenta and spin directions has been derived (including special forms);
- Polarization of spinning degree of freedom is found as the function of angular velocity and temperature. Minimum of polarization was observed.
- Presence of chiral phenomena in the model was proved.

Further research:

- Massless particles with continuous helicity;
- Systems with self-interaction;

The study is supported by the RSF grant No. 21-71-10066.


Thank you for attention!