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Spherically symmetric solution of Cotton gravity

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Plan of the talk

- An introduction to Cotton Gravity
- Spherically symmetric solution
- General spherically symmetric vacuum solution
- Velocity squared force
- Conclusions

General spherically symmetric solution of Cotton gravity

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An introduction to Cotton Gravity

Even though General Relativity can describe the gravitational dynamics of stars and planets successfully, it faces challenges when applied to bigger scales.

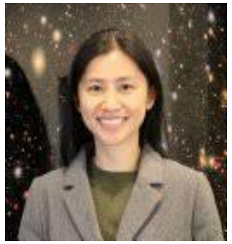
The idea behind Einstein's equations: Matter gravitates and creates a space-time curvature:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

There is no comprehensive theory of gravity, which is why we need to explore different models.

Cotton gravity is the theory which introduces modifications of the gravitational field equations by using the Cotton tensor as a curvature and gradients of energy-momentum tensor (Generalized angular momentum) as a source:

$$C^{\sigma}{}_{\nu\mu} = 8\pi G \left[\nabla_\mu T^\sigma_\nu - \nabla_\nu T^\sigma_\mu - \frac{1}{3}\left(\delta^\sigma_\nu \nabla_\mu T - \delta^\sigma_\mu \nabla_\nu T\right)\right] \qquad \textbf{(1)}$$



J. Harada, "*Emergence of the Cotton tensor for describing gravity,*" Phys. Rev. D (2021)

An introduction to Cotton Gravity

Cotton tensor (Émile Cotton; 1872-1950) can be expressed using various other tensors:

$$C_{\mu\nu\sigma} = 2\nabla_\alpha W^\alpha_{\mu\nu\sigma} = \nabla_\mu R_{\nu\sigma} - \nabla_\nu R_{\mu\sigma} - \frac{1}{6}\left(g_{\nu\sigma}\nabla_\mu R - g_{\mu\sigma}\nabla_\nu R\right) \qquad (2)$$

$$C_{\mu\rho\sigma} = -C_{\mu\sigma\rho}$$
$$C^\mu_{\mu\rho} = g^{\sigma\mu}C_{\sigma\mu\rho} = 0$$
$$C_{\mu\rho\sigma} = C_{\sigma\mu\rho} = C_{\rho\sigma\mu}$$

$\longrightarrow$ Satisfaction of these conditions ensures the preservation of the standard energy-momentum conservation law: $\nabla_\mu T^{\mu\nu} = 0$

Vacuum equations of cotton gravity: $\boldsymbol{C_{\mu\nu\sigma} = 0}$ (3)

Any solution of the Einstein equations is also a solution of Cotton gravity.

Spherically symmetric solution

Schwarzschild-type spherically symmetric solution to (3) is given by:

$$ds^2_{\mathrm{Sch}} = (1 - A)dt^2 - \frac{1}{(1 - A)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4)$$

Where

$$A(r) = \frac{2Gm}{r} + \gamma r - \Lambda r^2 \qquad (5)$$

An extra linear term which is absent in Einstein's gravity and has been successfully used to explain galaxy rotation curves without the need of dark matter.

γ and Λ are integration constants and they can be estimated using the radius of the observable universe (Hubble horizon).

General spherically symmetric vacuum solution

Generally, we can consider a case where $g_{tt}(r) = -1/g_{rr}(r)$ is not satisfied:

$$ds^2 = (1 - A)\, dt^2 - \frac{1}{(1 - B)}\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \quad \textbf{(6)}$$

If we express $A(r)$ by (5) and solve vacuum Cotton equations for second the unknown function, we get:

$$\begin{aligned} B(r) = {} & \frac{r^4\left(\Lambda r^3 - \gamma r^2 - 2Gm + r\right) C_2 - \left(3\gamma r^2 + 9Gm - 4r\right)\left(\Lambda r^3 - \gamma r^2 - 2Gm + r\right) C_1}{r\left(2r - \gamma r^2 - 6Gm\right)^2} + \\ & + \frac{\left(51\gamma^2\Lambda + 68\Lambda^2 + 60\gamma\Lambda^2 Gm\right) r^7 - \left(51\gamma^3 + 60\Lambda Gm\gamma^2\right) r^6}{17r\left(2r - \gamma r^2 - 6Gm\right)^2} - \\ & - \frac{\left(120\Lambda G^2 m^2\gamma + 42\gamma^2 Gm - 80\Lambda Gm - 68\gamma\right) r^4}{17r\left(2r - \gamma r^2 - 6Gm\right)^2} - \\ & - \frac{\left(180\Lambda G^2 m^2 + 208\gamma Gm\right) r^3 + \left(56Gm - 300\gamma G^2 m^2\right) r^2 - 360 G^3 m^3}{17r\left(2r - \gamma r^2 - 6Gm\right)^2} \end{aligned} \quad \textbf{(7)}$$

We are going to analyze this expression in a few different scenarios

General spherically symmetric vacuum solution

With no cosmological constant $(\Lambda = 0)$ metric functions take the form:

$$A_{\text{Harada}}(r) = \frac{2Gm}{r} + \gamma r$$

$$B_{\text{Harada}}(r) = \frac{r^4\left(2Gm + \gamma r^2 - r\right)C_2 + \left(2Gm + \gamma r^2 - r\right)\left(9Gm + 3\gamma r^2 - 4r\right)C_1}{r\left(6Gm + \gamma r^2 - 2r\right)^2} - \\ - \frac{51\gamma^3 r^6 - (42\gamma^2 Gm - 68\gamma)r^4 + 208\gamma Gmr^3 + (56Gm - 300\gamma G^2m^2)r^2 - 360G^3m^3}{17r\left(6Gm + \gamma r^2 - 2r\right)^2} \quad (7)$$

It is worth noting that at large distances, where $\frac{2mG}{r} \to 0$ and $\gamma r \le 1$, if we put integration constant $C_1 = \frac{48Gm}{17}$, the second function in (7) will obtain the form:

$$B_{\text{Harada}}(r) \approx \frac{2Gm}{r} + \gamma r + \frac{C_2}{4}r^2 \quad (8)$$

This is reminiscent of (5) and differs from it with the last term, C_2 is arbitrary.

General spherically symmetric vacuum solution

With no ordinary matter $(m = 0)$ metric functions take the form:

$$A_{\text{Dark}}(r) = \gamma r - \Lambda r^2$$
$$B_{\text{Dark}}(r) = \frac{r^3\left(\Lambda r^2 - \gamma r + 1\right)C_2 + \left(\Lambda r^2 - \gamma r + 1\right)(4 - 3\gamma r)C_1}{r(2 - \gamma r)^2} + \frac{r\left(3\gamma^2\Lambda r^3 - 3\gamma^3 r^2 + 4\Lambda^2 r^3 + 4\gamma\right)}{(2 - \gamma r)^2} \qquad \mathbf{(9)}$$

In the second function of (9) there are terms with the coefficient $\gamma\Lambda$. Since in Cotton gravity γ is related to the dark matter, this cross terms are indicating possible interactions between dark matter and dark energy.

General spherically symmetric vacuum solution

Now we consider the pure Schwarzschild case: $\gamma = \Lambda = 0$

$$A_{\mathrm{Sch}}(r) = \frac{2Gm}{r}$$

$$B_{\mathrm{Sch}}(r) = \frac{r^4(r-2Gm)C_2 + (4r-9Gm)(r-2Gm)C_1}{4r(r-3Gm)^2} + \frac{2Gm(45G^2m^2-7r^2)}{17r(r-3Gm)^2} \qquad (10)$$

Setting $C_1 = \frac{48Gm}{17}$ and $C_2 = 0$,

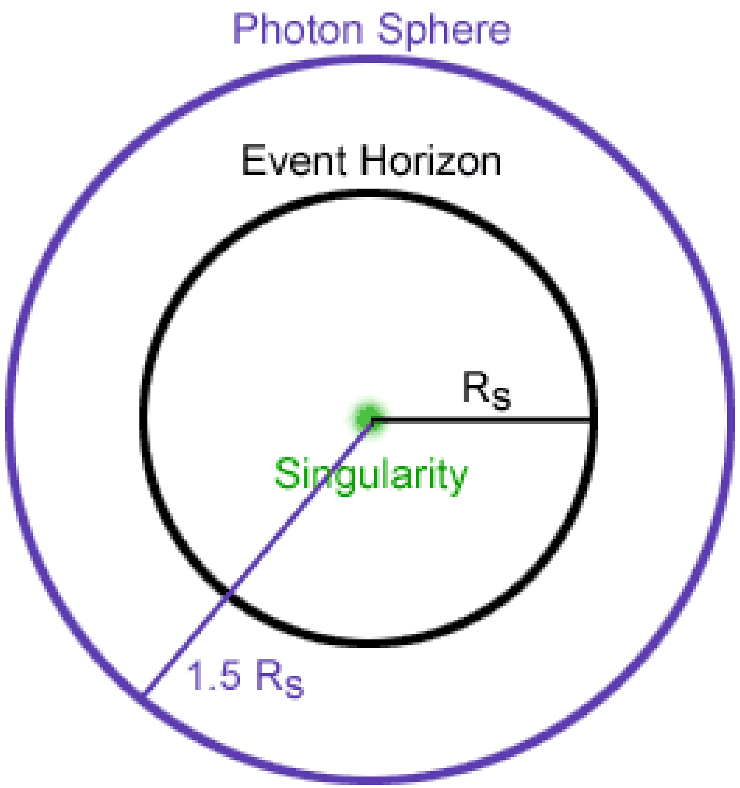
we get: $A_{Sch}(r) = B_{Sch}(r) = \frac{2Gm}{r}$

The appearance of the function $(3Gm - r)$ in the denominator is worth mentioning. It also presents in Ricci and Kretschmann scalars, which means there is already a singularity on the photon sphere.

General spherically symmetric vacuum solution

- Photon sphere is a surface around a black hole, where all orbiting photons are located.
- For non-rotating (Schwarzschild) black holes it is $\boldsymbol{R = 3Gm}$.

The derived metric exhibits an interesting property – the presence of singularities on the photon sphere, which could change our understanding of black holes and address problems concerning this matter.



Velocity squared force

Now we consider the weak gravity limit of the non-vacuum Cotton equations (1) for the pure Schwarzschild type metric (10):

$$ds^2 = [1 - A_{\mathrm{Sch}}(r)]dt^2 - [1 + B_{\mathrm{Sch}}(r)]dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{11}$$

The nonzero component of Cotton gravity equations (J. Harada, "*Cotton gravity and 84 galaxy rotation curves,*" Phys. Rev. D (2022)):

$$R_{00} + \frac{1}{6}R = -\frac{16\pi G}{3}\varepsilon \tag{12}$$

Using the metric (11)

$$A''_{Sch} + \frac{3A'_{Sch}}{2r} - \frac{1}{2}\left(B''_{Sch} + \frac{2B'_{Sch}}{r} + \frac{B_{Sch}}{r^2}\right) = -8\pi G\varepsilon \tag{13}$$

Assuming that the potentials $A_{Sch}(r)$ and $B_{Sch}(r)$ contain identical Newtonian terms, the solution to (13) will be:

$$A_{Sch} = \frac{2Gm}{r}; \qquad B_{Sch} = \frac{2Gm}{r} + \frac{C}{4}r^2; \qquad \varepsilon = \frac{9C}{32\pi G} \tag{14}$$

Velocity squared force

Appearance of the cosmological constant-like term $\frac{C}{4}r^2 = \frac{8\pi G\varepsilon}{9}r^2$ in one of the metric functions eventually leads to the velocity dependent term in the geodesic equation.

The geodesic equations under the assumptions of a static spacetime and small curvature:

$$\frac{d^2x^i}{dt^2} \approx -\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\Gamma^i{}_{\mu\nu} \approx -\Gamma^i{}_{00} - \Gamma^i{}_{jk} v^j v^k$$

3-velocity, which is normally ignored in Newton's approximation where $g'_{00} \approx g'_{rr}$

In the spherically symmetric case of Cotton gravity, for the radial acceleration we get:

$$a_r \approx \frac{1}{2}\left(g'_{00} - g'_{rr}v_r^2\right) \approx -\frac{Gm}{r^2} + \frac{32\pi G\varepsilon}{9} r\, v_r^2 \qquad (15)$$

This is the repulsive long-range correction to the Newton's gravitational force, which could suggest the alternative explanation for the observed accelerated expansion of the universe.

K. Loeve, K. S. Nielsen and S. H. Hansen, *"Consistency analysis of a Dark Matter velocity dependent force as an alternative to the Cosmological Constant,"* Astrophys. J. (2021).

Summary of the results

- We have obtained the general spherically symmetric solution to the vacuum equations of Cotton gravity and examined its various features.
- Within the framework of Cotton gravity, black hole solutions may exhibit significant deviations from the predictions of Einstein's theory. Singularities appear on the photon sphere, which prevents geodesics to be continued up to the central singularity at $\boldsymbol{r = 0}$.
- Two main parameters in the spherical symmetric solution (integration constants) can be associated with the Hubble's horizon.
- The velocity squared force within the Cotton gravity framework is examined. The new metric allows the appearance of velocity-squared repulsive long-range corrections to Newton's gravitational force.

Thank you
for your attention