Skyrme–Chern-Simons densities in all dimensions

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Definition of Chern-Simons density

The CS density in odd d dimensions descends from the Chern-Pontryagin (CP) density in even $d+1$ dimensions

$$\varrho \equiv \Omega_{\rm CP}^{(d+1)} = \varepsilon_{i_1 i_2 i_3 i_4 \dots i_d i_{d+1}} \operatorname{Tr} F_{i_1 i_2} F_{i_3 i_4} \dots F_{i_d i_{d+1}} \,, \tag{1}$$

which is **gauge invariant** by definition and is **total divergence**

$$\Omega_{\rm CP}^{(d+1)} = \partial_i \Omega_i^{(d+1)} \quad , \quad i = 1, 2, \dots, d+1 \tag{2}$$

The CS density is defined as the $(d+1)$-th component of $\Omega_i^{(d+1)}$

$$\Omega_{\rm CS}^{(d)}[A_\mu, F_{\mu\nu}] \stackrel{\rm def.}{=} (\Omega_{\rm CP}^{(d+1)})_{i=d+1} \,, \tag{3}$$

Since $\Omega_{\rm CP}^{(d+1)}$ is a "curl" defined in terms of the totally antisymmetric tensor $\varepsilon^{i_1 i_2 \dots i_{d+1}}$, fixing one component leads to a descent by one dimension, *i.e.*, $\Omega_{\rm CS}^{(d)}$ is a scalar in d dimensions, expressed in terms of the gauge connection A_μ and the curvature $F_{\mu\nu}$ with coordinates x_μ

Note that CS density (3) is **not explicitly gauge variant**.

Gauge transformation of CS density (infinitesimal)

While $\Omega^{(d)}_{\rm CS}$ is **not explicitly gauge variant** it is **effectively gauge variant**: It is gauge invariant **up to a total divergence**

Consider an infinitesimal gauge transformation

$$\Omega_i^{(d+1)} \stackrel{g}{\to} \Omega_i^{(d+1)} + \delta\Omega_i^{(d+1)} \,. \tag{4}$$

Since $\varrho = \partial_i\Omega_i^{(d+1)}$ is gauge invariant, *i.e.*, $\delta\varrho = 0$, it follows that

$$\delta(\partial_i\Omega_i^{(d+1)}) = 0 \qquad \Rightarrow \qquad \partial_i(\delta\Omega_i^{(d+1)}) = 0 \tag{5}$$

$\delta\Omega_i^{(d+1)}$ can formally be expressed as

$$\delta\Omega_i^{(d+1)} = \varepsilon_{ijk_1k_2\dots k_{d-1}}\,\partial_j\Lambda_{k_1k_2\dots k_{d-1}}\,, \tag{6}$$

where $\Lambda_{k_1k_2\dots k_{d-1}}[A_i, F_{ij}]$ is a totally antisymmetric tensor.

From the definition (3) of the CS density, (6) implies the following infinitesimal transformation of the CS density

$$\delta\Omega_{\rm CS}^{(d)} = \delta\Omega_{i=d+1}^{(d+1)} = \varepsilon_{(d+1)\mu\nu_1\nu_2\dots\nu_{d-1}}\,\partial_\mu V_{\nu_1\nu_2\dots\nu_{d-1}} \tag{7}$$

defined on the space with the d-dimensional coordinates x_μ.

It follows from (7) that under an infinitesimal gauge transformation $g(x_\mu)$, the CS density $\Omega_{\rm CS}^{(d)}$ transforms as

$$\Omega_{\rm CS}^{(d)} \stackrel{g}{\rightarrow} \Omega_{\rm CS}^{(d)} + \varepsilon_{\mu\nu_1\nu_2\dots\nu_{d-1}}\,\partial_\mu V_{\nu_1\nu_2\dots\nu_{d-1}}\,, \tag{8}$$

meaning that the CS density is *gauge invariant up to a total divergence*.

One concludes that the action of $\Omega_{\rm CS}^{(d)}$, namely its volume integral

$$\int d^d x\ \Omega_{\rm CS}^{(d)} \stackrel{g}{\rightarrow} \int d^d x\ \Omega_{\rm CS}^{(d)}$$

remains invariant under the action of g, resulting in gauge invariant Euler-Lagrange equations.

Examples of (usual) CS densities

For $d = 3$ and $d = 5$,

$$\Omega_{\rm CS}^{(1)} = \varepsilon_{\mu\nu\lambda} \mathrm{Tr}\, A_\lambda \left(F_{\mu\nu} - \frac{2}{3} F_i F_j \right) .$$

$$\Omega_{\rm CS}^{(2)} = \mathrm{Tr}\, A \wedge \left(F \wedge F - F \wedge A \wedge A + \frac{2}{5} A \wedge A \wedge A \wedge A \right) .$$

and for $d = 7$

$$\begin{aligned} \Omega_{\rm CS}^{(3)} &= \mathrm{Tr}\, A \wedge \left(F \wedge F \wedge F - \frac{4}{5} F \wedge F \wedge A \wedge A - \frac{2}{5} F \wedge A \wedge F \wedge A \right. \\ &\qquad \left. + \frac{4}{5} F \wedge A \wedge A \wedge A \wedge A - \frac{8}{35} A \wedge A \wedge A \wedge A \wedge A \wedge A \right) \\ \tilde{\Omega}_{\rm CS}^{(3)} &= \mathrm{Tr}\, A \wedge \left(F - \frac{2}{3} A \wedge A \right) \cdot (\mathrm{Tr}\, F \wedge F) \end{aligned}$$

$\tilde{\Omega}_{\rm CS}^{(3)}$ corresponding to **double trace** definition of CP density

Gauge transformation of CS density (global)

As shown above in all odd dimensions CS densities transform as

$$\Omega_{\rm CP} \to \Omega_{\rm CP} + \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}$$

In $d = 3$ and 5, such explicit expressions for global gauge transformations featuring the element $\alpha_\mu = \partial_\mu g\, g^{-1}$ of the gauge group element $g(x)$ are

$$\Omega^{(2)}_{\rm CS} \to \tilde{\Omega}^{(2)}_{\rm CS} = \Omega^{(2)}_{\rm CS} - \frac{2}{3}\varepsilon_{\lambda\mu\nu}\,\mathrm{Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu - 2\varepsilon_{\lambda\mu\nu}\,\partial_\lambda \mathrm{Tr}\,\alpha_\mu\, A_\nu$$

$$\Omega^{(3)}_{\rm CS} \to \tilde{\Omega}^{(3)}_{\rm CS} = \Omega^{(3)}_{\rm CS} - \frac{2}{5}\,\varepsilon_{\lambda\mu\nu\rho\sigma}\mathrm{Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu\alpha_\rho\alpha_\sigma$$

$$+2\,\varepsilon_{\lambda\mu\nu\rho\sigma}\,\partial_\lambda \mathrm{Tr}\,\alpha_\mu\left[A_\nu\left(F_{\rho\sigma} - \frac{1}{2}A_\rho A_\sigma\right) + \left(F_{\rho\sigma} - \frac{1}{2}A_\rho A_\sigma\right)A_\nu - \frac{1}{2}\,A_\nu\,\alpha_\rho\,A_\sigma - \alpha_\nu\,\alpha_\rho\,A_\sigma\right]$$

in which α_μ can be encoded with a "winding number".

Higgs–Chern-Pontryagin (HCP) densities

The n-th Chern-Pontryagin (CP) density in $2n$ dimensions is

$$\mathcal{C}^{(n)}[F] = \partial_I \Omega_I^{(n)}[A, F]\ , \quad I = 1, 2, \dots, 2n$$

Subjecting it to descent to D residual residual dimensions over a compact coset co-dimension K^{2n-D}, its volume integral (up to the "angular volume" of K^{2n-D})

$$\int_{\mathbf{R}^D \times K^{2n-D}} \mathcal{C}^{(n)} \simeq \int_{\mathbf{R}^D} \partial_i \Omega_i^{(n,D)}[F, \Phi] \quad \text{odd} \quad D \tag{9}$$

$$\simeq \int_{\mathbf{R}^D} \partial_i \Omega_i^{(n,D)}[A, F, \Phi] \quad \text{even} \quad D \tag{10}$$

The integrand in (9)-(10) is a total divergence in the residual space $\mathbb{R}^D$ with $i = 1, 2, \dots, D$.
The residual gauge group and the rank of the scalar (Higgs) field Φ depend on the choice of K^{2n-D}.

The notations in (9) and (10) imply that $\Omega_i^{(n,D)}$ are gauge invariant and gauge variant, respectively for odd and even D.

In even residual dimensions D, $\partial_i\Omega_i^{(n,D)}$ consists of the usual CP density defined exclusively by the gauge field, plus a Higgs dependant term.

Denoting the integrands in (9) and (10)

$$\mathcal{C}_{\rm HCP}^{(n,D)} \stackrel{\rm def.}{=} \partial_i\Omega_i^{(n,D)} \ , \quad i=1,2,\dots,D \tag{11}$$

gives the definition of the Higgs–Chern-Pontryagin (HCP) densities on $\mathbb{R}^D$, where now D can be both even and odd!

The HCP densities (11) are both **gauge invariant** and are manifestly **total-divergence**

It is therefore possible to define the Higgs–Chern-Simons (HCS) densities *via* the one-step descent described in the usual case.

Examples of Higgs–Chern-Pontryagin (HCP) densities

- $n = 3\ ,\quad D = 5$

$$\begin{aligned}\mathcal{C}_{\rm HCP}^{(3,5)} &= \varepsilon_{ijklm}\mathrm{Tr}\ F_{ij}\,F_{kl}\,D_m\Phi = \partial_m\Omega_m^{(3,5)}\ ,\\ \Omega_m^{(3,5)} &= \varepsilon_{ijklm}\,\mathrm{Tr}\ F_{ij}\,F_{kl}\,\Phi\end{aligned}$$

- $n = 3\ ,\quad D = 4$

$$\begin{aligned}\mathcal{C}_{\rm HCP}^{(3,4)} &= \varepsilon_{ijkl}\,\mathrm{Tr}\,\Gamma_5\ (S\,F_{ij}F_{kl} + 2\,D_i\Phi\,D_j\Phi\,F_{kl}) = \partial_i\Omega_i^{(3,4)}\\ \Omega_i^{(3,4)} &= \varepsilon_{ijkl}\mathrm{Tr}\,\Gamma_5\Big[-2\eta^2 A_j\left(F_{kl} - \frac{2}{3}\,A_kA_l\right) +\\ &\qquad + (\Phi\,D_j\Phi - D_j\Phi\,\Phi)\,F_{kl}\Big]\end{aligned}$$

$$S = -\left(\eta^2 \mathbb{1} + \Phi^2\right)\ ,\quad \text{and}\quad \Gamma_5\quad \text{is symbolic}$$

- $n = 4\ ,\quad D = 7$

$$\mathcal{C}_{\rm HCP}^{(4,7)} = \varepsilon_{ijklmnp}\mathrm{Tr}\ F_{ij}\,F_{kl}\,F_{mn}\,D_p\Phi = \partial_p\Omega_p^{(4,7)}$$
$$\Omega_p^{(4,7)} = \varepsilon_{ijklmnp}\,\mathrm{Tr}\ F_{ij}\,F_{kl}\,F_{mn}\,\Phi$$

- $n = 4\ ,\quad D = 6$

$$\mathcal{C}_{\rm HCP}^{(4,6)} = \varepsilon_{ijklmn}\,\mathrm{Tr}\,\Gamma_7\left[S\,F_{ij}F_{kl}F_{mn} + 2\,F_{ij}F_{kl}D_m\Phi D_n\Phi + \right.$$
$$\left. + F_{ij}D_m\Phi F_{kl}D_n\Phi\right] = \partial_i\Omega_i^{(4,6)}$$
$$\Omega_i^{(4,6)} = \varepsilon_{ijklmn}\,\mathrm{Tr}\,\Gamma_7\,A_j\left[\left(F_{kl}F_{mn} - F_{kl}A_mA_n + \frac{2}{5}A_kA_lA_mA_n\right) + \right.$$
$$\left. + D_j\Phi\left(\Phi F_{kl}F_{mn} + F_{kl}\Phi F_{mn} + F_{kl}F_{mn}\Phi\right)\right]$$

The Higgs–Chern-Simons (HCS) densities

The HCS density in any odd or even dimension $d = D - 1$ follows by setting the index $i = D$, thus achieving the one step descent from the HCP to the HCS density, in each case.

The salient feature of all $\Omega_i^{(n,D)}$ for all **even** D is that the leading term independent of the Higgs scalar Φ is the usual CP density in that dimension. Thus the leading term in any odd dimension d is the usual CS density in that dimension.

Since the HCP density is both gauge invariant and total-divergence, it follows that the HCS density is **gauge invariant up to a total-divergence term**, as in the usual CS case.

Skyrme–Chern-Simons (HCS) densities

The definition of CS and HCS densities was made by the one-step descent from the respective CP and HCP densities

HCP densities descend from CP densities via (compact) coset-space dimensional reduction: this results in the HCP being both **gauge invariant** and **total divergence**, like the CP

In contrast to the gauged Higgs case

$$CP \to HCP$$

such a procedure is absent in the gauged Skyrme case!

In the case of gauged Skyrme systems in D dimensions, a prescription for defining a density which is both **gauge invariant** and **total divergence** is needed

Skyrme Sigma models

- Skyrme sigma models are the $O(D+1)$ sigma models in **all** D-dimensions
- They are defined in terms of a scalar field ϕ^a, $a = 1, 2, \ldots, D+1$ subject to the constraint

$$|\phi^a|^2 = 1$$

- The Rotation invariant density

$$\varrho_0^{(D)} = \varepsilon_{i_1 i_2 \ldots i_D} \varepsilon^{a_1 a_2 \ldots a_D a_{D+1}} \partial_{i_1} \phi^{a_1} \, \partial_{i_2} \phi^{a_2} \ldots \partial_{i_D} \phi^{a_D} \, \phi^{a_{D+1}} \tag{12}$$

is *essentially total divergence* in the sense that when subjected to the variational principle,
with the constraint taken into account, it does not yield non-trivial equations of motion

- In a parametrisation of ϕ^a that satisfies the sigma constraint $\varrho_0^{(D)}$ becomes *exlpicitly total divergence e.g.*, using the coordinates on S^D, with $f^{(1)}$, $f^{(2)}, \dots, f^{(D-1)}$ being the 'polar' fuction and g the 'azimuthal' function,

$$\begin{aligned} \frac{1}{D!}\varrho_0^{(D)} &= \varepsilon_{i_1 i_2 \dots i_D}\, \partial_{i_1} f^{(1)}\, \partial_{i_2} f^{(2)} \dots, \partial_{i_{D-1}} f^{(D-1)}\, \partial_{i_D} g\, \cdot \\ &\quad \cdot \sin^{D-1} f^{(1)} \sin^{D-2} f^{(2)} \dots \sin f^{(D-1)} \sin g \\ &\stackrel{\text{def.}}{=} \boldsymbol{\nabla} \cdot \boldsymbol{\omega}^{(D)} \end{aligned}$$

  which is manifestly total divergence.
  Note that there is an increasing number of choices in the definition of $\boldsymbol{\omega}$, with increasing dimension.
- When the D-dimensional space is $\mathbb{R}^D$, then $\varrho_0^{(D)}$ is a winding number (up to the angular volume), and is a topological charge density

Gauged Skyrme models

The prescription for gauging the Skyrme scalar $O(D+1)$ Skyrme scalar in D dimensions is

$$\begin{aligned} D_i\phi^\alpha &= \partial_i\phi^\alpha + (A_i\phi)^\alpha \ , \quad \alpha = 1,2,\dots,D \\ D_i\phi^{D+1} &= \partial_i\phi^{D+1} \end{aligned}$$

A_i being the $SO(D)$ gauge connection

$$A_i^{\alpha\beta} = -A_i^{\beta\alpha} \ .$$

This includes all possible contractions of $SO(D)$ algebra.

The winding number density ϱ_0 (12) is **gauge variant** and **total-divergence**.

Replacing all partial derivatives in ϱ_0 with covariant derivatives results in the density

$$\varrho_G^{(D)} = \varepsilon_{i_1 i_2 \dots i_D}\varepsilon^{a_1 a_2 \dots a_D a_{D+1}} \, D_{i_1}\phi^{a_1} \, D_{i_2}\phi^{a_2} \dots \, D_{i_D}\phi^{a_D} \, \phi^{a_{D+1}}$$

which is **gauge invariant** but is **not total-divergence**!

The required density playing the roles of the CP and HCP densities must like them be **both gauge-invariant and total-divergence**

To this end, one calcualtes the difference of ϱ_G and ϱ_0

$$\varrho_G^{(D)} - \varrho_0^{(D)} = \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}[A, \phi] - W[F, D\phi] \tag{13}$$

where $\boldsymbol{\Omega}[A, \phi]$ is **gauge variant** and $W[F, D\phi]$ is **gauge invariant**

Note: In practice, this is done by removing a total-divergence in (13) until a gauge-invariant quantity W is isolated **in each case**.

Then, rearranging the terms in (13) gives the required density

$$\varrho^{(D)} \stackrel{\text{def.}}{=} \varrho_G^{(D)} + W[F, D\phi] \tag{14}$$

$$= \varrho_0^{(D)} + \boldsymbol{\nabla} \cdot \boldsymbol{\Omega}[A, \phi] \tag{15}$$

which is **both gauge-invariant and total-divergence**.

Skyrme–Chern-Pontryagin (SCP) densities

Let us flippantly call the density $\varrho^{(D)}$ defined by (14)-(15) as the Skyrme–Chern-Pontryagin (SCP) density

Definition (14), expressed exclusively in terms of **gauge invariant** quantities, is employed in stating Bogomoln'yi type inequalities, thus qualifying it as an **"energy density lower bound"**

Definition (15) clearly expresses the departure of this "bound" from the winding number, or **"baryon number"** due to the introduction of the gauge field to the Skyrme system, and like the latter is a **total-divergence** enabling the evaluation of it by a *surface integral*

Most importantly here, the definition (15) enables the passage to the Skyrme–Chern-Simons (SCS) density by the one-step reduction employed in passing from CP$\rightarrow$CS and HCP$\rightarrow$HCS

Examples of Skyrme–Chern-Pontryagin (SCP) densities

It is sufficient to present $\varrho^{(D)}$ in D dimensions for $SO(D)$ gauging.

The gauge connection $A_i^{\alpha\beta}$ in the algebra of $SO(D)$ can then be subjected to *contractions* to all possible subalgebras

In $D = 2$ with $SO(2)$ gauging:

$$\begin{aligned} \varrho^{(2)}_{SO(2)} &= \varrho^{(2)}_G + \varepsilon_{ij}\phi^3 F_{ij} \\ &= \varrho^{(2)}_0 + 2\varepsilon_{ij}\partial_i(\phi^3 A_j) \end{aligned}$$

In $D = 3$ with $SO(3)$ gauging

$$\begin{aligned} \varrho^{(3)}_{SO(3)} &= \varrho^{(3)}_G + \varepsilon_{ijk}\phi^4 \varepsilon^{\alpha\beta\beta'} F_{ij}^{\beta\beta'} D_k \phi^\alpha \\ &= \varrho^{(3)}_0 + \varepsilon_{ijk}\partial_k \phi^\alpha \varepsilon^{\alpha\beta\beta'} \left[2A_i^{\beta\beta'} \partial_j \phi^4 - \phi^4 F_{ij}^{\beta\beta'} \right] \end{aligned}$$

In this case the $SO(3)$ gauge connection can be contracted to $SO(2)$

In the $D=4$ with $SO(4)$ gauging

$$
\begin{aligned}
\varrho^{(4)}_{SO(4)} &= \varrho^{(4)}_{G} + \varepsilon_{ijkl}\varepsilon^{\alpha\beta\gamma\delta}\phi^5[3F^{\gamma\delta}_{kl}\, D_i\phi^{\alpha} D_j\phi^{\beta} + \frac{1}{4}(\phi^5)^2 F^{\alpha\beta}_{ij} F^{\gamma\delta}_{kl}] \\
= \quad & \varrho^{(4)}_0 + \frac{3}{2}\varepsilon_{ijkl}\varepsilon^{\alpha\alpha'\beta\beta'}\partial_i\{\phi^5(1-\frac{1}{3}(\phi^5)^2)A^{\alpha\alpha'}_j[\partial_k A^{\beta\beta'}_l + \frac{2}{3}(A_k A_l)^{\beta\beta'}]\} \\
& +6\varepsilon_{ijkl}\partial_i\Big\{\phi^5\varepsilon^{\alpha\beta\gamma\gamma'}[\phi^{\alpha}(A_j\phi)^{\beta} - \partial_k\phi^{\alpha}\phi^{\beta}](A_k A_l)^{\gamma\gamma'} \\
& \qquad -\frac{1}{4}\phi^5(1-(\phi^5)^2)\varepsilon^{\alpha\alpha'\beta\beta'}A^{\alpha\alpha'}_j\partial_k A^{\beta\beta'}_l\Big\}
\end{aligned}
$$

In this case the $SO(4)$ gauge connection can be contracted to $SO(2)\times SO(2)$, $SO(3)$ and $SO(2)$

For further illustration, list $\varrho^{(3)}$, $\varrho^{(4)}$ and $\varrho^{(5)}$ all three for $SO(2)$ gauging

$$
\begin{aligned}
\varrho^{(3)}_{SO(2)} &= \varrho^{(3)}_{G} + \varepsilon_{ijk}\varepsilon^{AB}(F_{ij}\phi^B D_k\phi^A) \\
&= \varrho^{(3)}_{0} + \varepsilon_{ijk}\varepsilon^{AB}\partial_i(A_j\,\partial_k\phi^A\,\phi^B)
\end{aligned}
$$

$$
\begin{aligned}
\varrho^{(4)}_{SO(2)} &= \varrho^{(4)}_{G} + 2\,\varepsilon_{ijkl}\varepsilon^{ABC}\left(F_{ij}\,\partial_k\,\phi^A\,\partial_l\,\phi^B\,\phi^C\right) \\
&= \varrho^{(4)}_{0} + 4\,\varepsilon_{ijkl}\varepsilon^{ABC}\,\partial_i\left(A_j\partial_k\,\phi^A\,\partial_l\,\phi^B\,\phi^C\right)
\end{aligned}
$$

$$
\begin{aligned}
\varrho^{(5)}_{SO(2)} &= \varrho^{(5)}_{G} + \frac{1}{2}\,F_{ij}\,\varepsilon_{ijklm}\varepsilon^{ABCD}\left(\partial_k\phi^A\partial_l\phi^B\partial_m\phi^C\phi^D\right) \\
&= \varrho^{(5)}_{0} + \varepsilon_{ijklm}\varepsilon^{ABCD}\,\partial_i\left[A_j\left(\partial_k\phi^A\partial_l\phi^B\partial_m\phi^C\phi^D\right)\right]
\end{aligned}
$$

The Skyrme–Chern-Simons (SCS) densities

To define a SCS density in $d = D - 1$ dimensions the definition (15) of the SCP density $\varrho^{(D)}$ is employed

To express this density as a **total-divergence** a constraint compliant paramertisation of ϱ_0 must be used, such that

$$\varrho_0 = \partial_i \omega_i$$

$$\varrho^{(D)} = \partial_i \omega_i^{(D)} + \partial_i \Omega_i[A, \phi] \qquad (16)$$

Finally the one-step descent resulting in the passage SCP→SCS is achieved by setting the index $i = D$ in (16)

$$\Omega_{\rm SCS}^{(D-1)} \stackrel{\rm def.}{=} \omega_{i=D}^{(D)} + \Omega_{i=D}[A, \phi]$$

In three dimensional Minkowski space, the $SO(4)$ SCS density is

$$
\begin{aligned}
{}^{SO(4)}\Omega^{(2,1)}_{\mathrm{SCS}} &= \omega^{(4)}_4 + \\
&+\frac{3}{2}\varepsilon_{\mu\nu\lambda}\varepsilon^{\alpha\alpha'\beta\beta'}\{\phi^5(1-\frac{1}{3}(\phi^5)^2)A^{\alpha\alpha'}_\lambda[\partial_\mu A^{\beta\beta'}_\nu + \frac{2}{3}(A_\mu A_\nu)^{\beta\beta'}]\} \\
&\quad +6\varepsilon_{\mu\nu\lambda}\partial_i\Big\{\phi^5\varepsilon^{\alpha\beta\gamma\gamma'}[\phi^\alpha A_\lambda\phi^\beta - \partial_\lambda\phi^\alpha\phi^\beta](A_\mu A_\nu)^{\gamma\gamma'} \\
&\qquad -\frac{1}{4}\phi^5(1-(\phi^5)^2)\varepsilon^{\alpha\alpha'\beta\beta'}A^{\alpha\alpha'}_\lambda\partial_\mu A^{\beta\beta'}_\nu\Big\}
\end{aligned}
$$

and the $SO(2)$ SCS density is

$$
{}^{SO(2)}\Omega^{(2,1)}_{\mathrm{SCS}} = \omega^{(4)}_4 + \varepsilon_{\mu\nu\lambda}\varepsilon^{ABC}\left(A_\lambda\partial_\mu\,\phi^A\,\partial_\nu\,\phi^B\,\phi^C\right)
$$

In four dimensional Minkowski space, the $SO(2)$ SCS density is

$$
{}^{SO(2)}\Omega^{(3,1)}_{\mathrm{SCS}} = \omega^{(5)}_5 + \varepsilon_{\mu\nu\tau\lambda}\varepsilon^{ABCD}\left(A_\lambda\partial_\mu\,\phi^A\,\partial_\nu\,\phi^B\,\partial_\tau\,\phi^C\,\phi^D\right)
$$

For the $SO(2) \times SO(2)$ case, with the two Abelian fields A_i and B_i and using the constraint compliant parametrisation

$$\phi^a = \begin{pmatrix} \phi^\alpha \\ \phi^A \\ \phi^5 \end{pmatrix} = \begin{pmatrix} \sin f(x_\mu) \sin g(x_\mu) \ n^\alpha \\ \sin f(x_\mu) \cos g(x_\mu) \ m^A \\ \cos f(x_\mu) \end{pmatrix}$$

with

$$n^\alpha = \begin{pmatrix} \cos \psi(x_\mu) \\ \sin \psi(x_\mu) \end{pmatrix} , \quad m^A = \begin{pmatrix} \cos \chi(x_\mu) \\ \sin \chi(x_\mu) \end{pmatrix} ,$$

$$\begin{aligned} \Omega_{\rm SCS} = \varepsilon^{\mu\nu\lambda} \Big\{ & -2 \left(\cos f - \frac{1}{3} \cos^3 f \right) (\partial_\lambda \sin^2 g)\, \partial_\mu \psi \, \partial_\nu \chi \\ & + \frac{1}{3} \cos^3 f \, (A_\lambda G_{\mu\nu} + B_\lambda F_{\mu\nu}) \\ & - A_\mu B_\nu \cos f \left[\partial_\lambda (\sin^2 f \sin^2 g) - \partial_\lambda (\sin^2 f \cos^2 g) \right] \\ & + 2 \cos f \left[A_\lambda \, \partial_\mu (\sin^2 f \cos^2 g) \, \partial_\nu \chi + B_\lambda \, \partial_\mu (\sin^2 f \sin^2 g) \, \partial_\nu \psi \right] \Big\} \end{aligned}$$

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Comments: time permitting

- Why Skyrme and not "Higgs" (*i.e.*, symmetry breaking)?
- Can this prescription for constructing SCP be applied to constructing HCP as an **alternative** to employing the method of dimensional reduction described above?

  Yes, **but...**
- Q: What is the motvation of employing SCS actions?

  Ans.: Negative slopes of E $vs.Q_e$ and E $vs.J$ and "baryon number" dissipation in odd dimensions (only)
- Have such solitons been constructed?

  Yes, with Francisco Navarro-Lerida and Eugen Radu