Gauged Skyrmions

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Thanks to my collaborators:
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Outline

- Skyrme model
- Rational maps and Multiskyrmions
- U(1) gauged Skyrme model
- Multisolitons in the U(1) gauged Skyrme model
- Summary

Skyrme model

- **QCD:** $$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}_i [i\gamma^\mu D_\mu - m\delta_{ij}]\psi_j$$

- **Low energy meson theory:**

$$L = \frac{f_\pi^2}{4} \mathrm{Tr}\,(\partial_\mu U \partial^\mu U) + \dots$$

Tony Hilton Royle Skyrme

Skyrmes' motivations (1962):

- The idea of unifying bosons and fermions in a common framework
- Consideration of localised field configurations instead of point-like particles
- The desire to eliminate fermions from a fundamental formulation of theory

Problems with Skyrme model:

- The binding energy is too high
- Clustering of light nuclei?
- p/n mass splitting? $m_n = 939,556\ MeV, \quad m_p = 938,272\ MeV$

Skyrme family

(J. Verbaarschot (1986),I Bogolubsky (1989), T.Tchrakian, W.Zakrzewski & R.Leese (1990))

- **(2+1)-dim: Baby Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\left(\partial_\mu\phi\times\partial_\nu\phi\right)^2 - V(\phi)$$

$$\phi: S^2 \to S^2; \quad \phi_\infty = (0,0,1)$$

Standard choice: $V(\phi) = \mu^2(1-\phi_3)$

$$Q \in \mathbb{Z} = \pi_2(S^2)$$

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2}\phi\cdot(\partial_1\phi\times\partial_2\phi)d^2x$$

- **(3+1)-dim: Skyrme model**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\left(\partial_\mu\phi\times\partial_\nu\phi\right)^2 - V(\phi)$$

$$\phi: S^3 \to S^3; \quad \phi_\infty = (0,0,0,1)$$

$$R_\mu = \partial_\mu U U^\dagger; \quad U = \phi_0\mathbb{I} + i\sigma^a\cdot\phi^a$$

$$\mathcal{L} = -\,\mathrm{Tr}\left\{\frac{1}{2}(R_\mu R^\mu) + \frac{1}{16}([R_\mu,R_\nu][R^\mu,R^\nu]) + \mu^2(U-\mathbb{I})\right\}$$

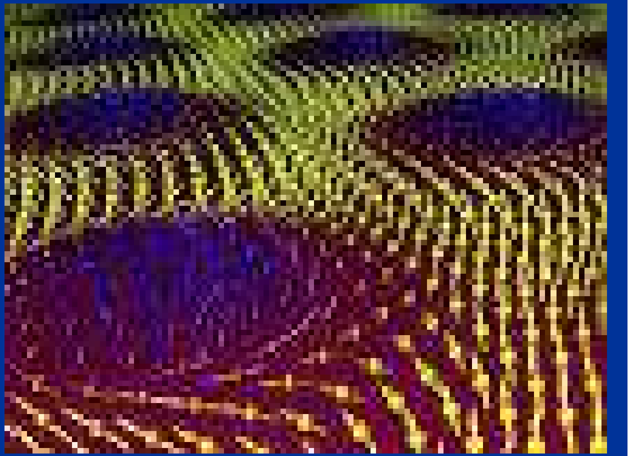
↓

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi^a)^2 - \tfrac{1}{4}[(\partial_\mu\phi^a\partial_\nu\phi^a)^2 - (\partial_\mu\phi^a)^4] + \mu^2\ (1-\phi^3)$$

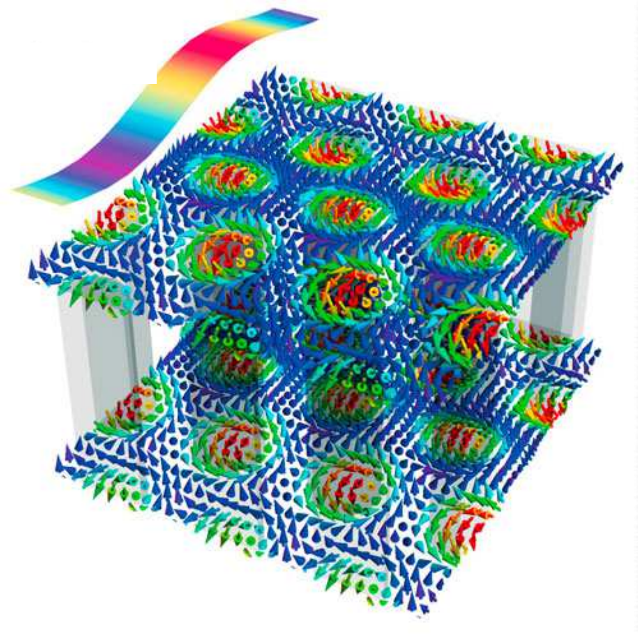
$$Q \in \mathbb{Z} = \pi_3(S^3)$$

$$Q = \frac{1}{24\pi^2}\,\mathrm{Tr}\int_{\mathbb{R}^3}\varepsilon_{ijk}R_iR_jR_k d^3x$$

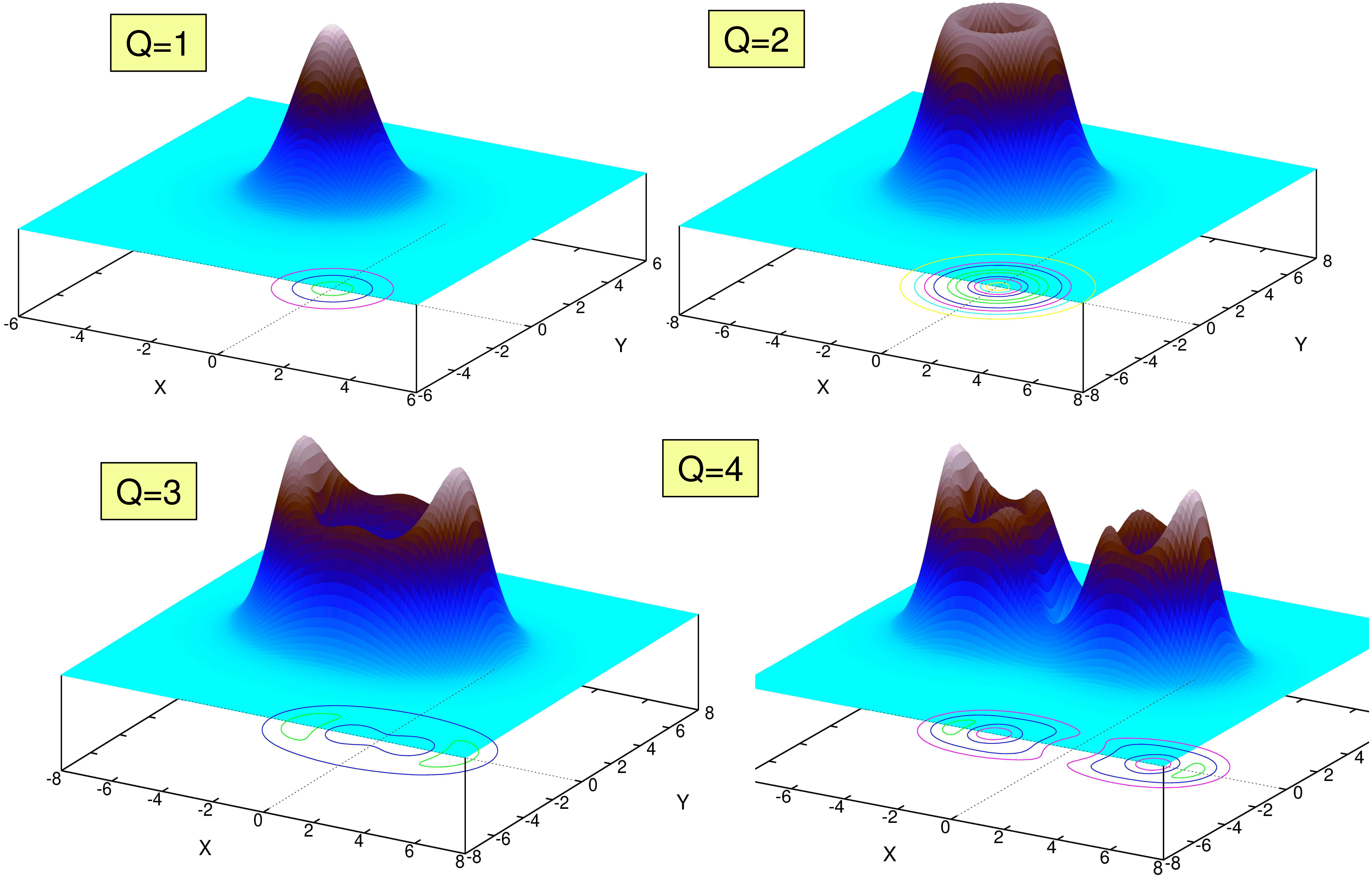
Baby Skyrme model: Applications



- A Heisenberg-type model of interacting spins
- A model of the topological quantum Hall effect
- Elementary excitations in quantum Hall magnets
- Chiral magnetic structures
- A model of ferromagnetic planar structures
- Applications in future development of data storage technologies
- Models of condensed matter systems with intrinsic and induced chirality

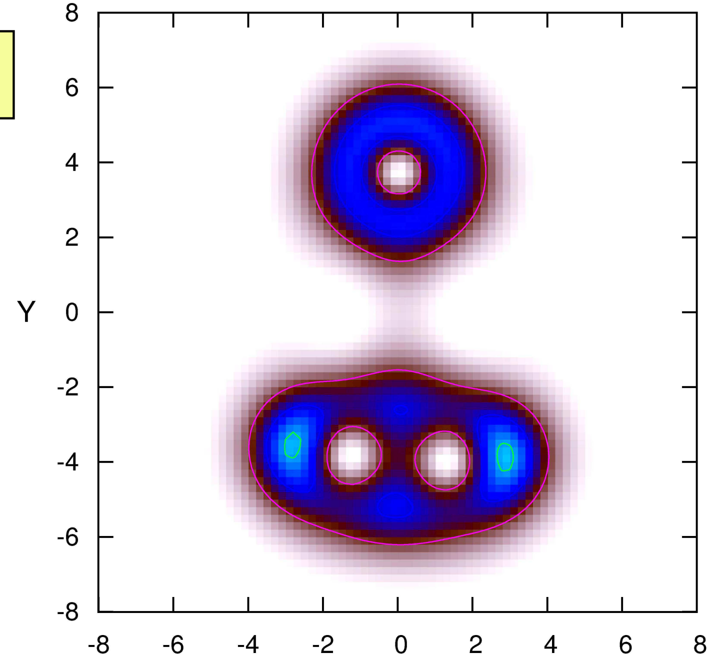


Baby Skyrme model: solitons

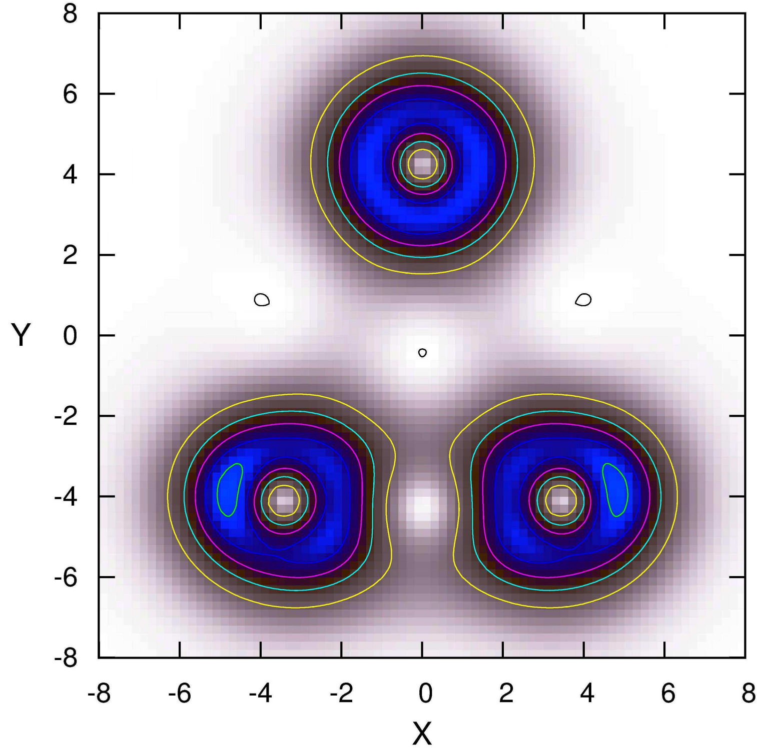


Baby Skyrme model: solitons

Q=5



Q=6



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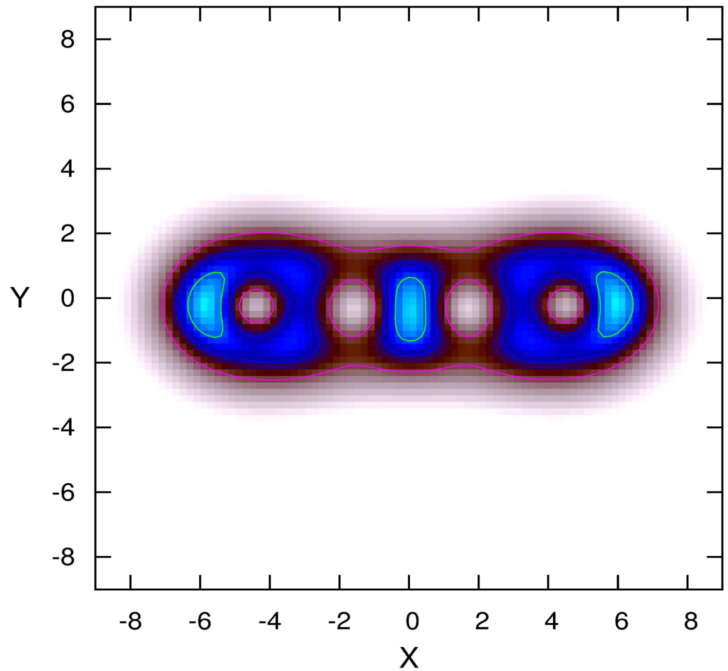

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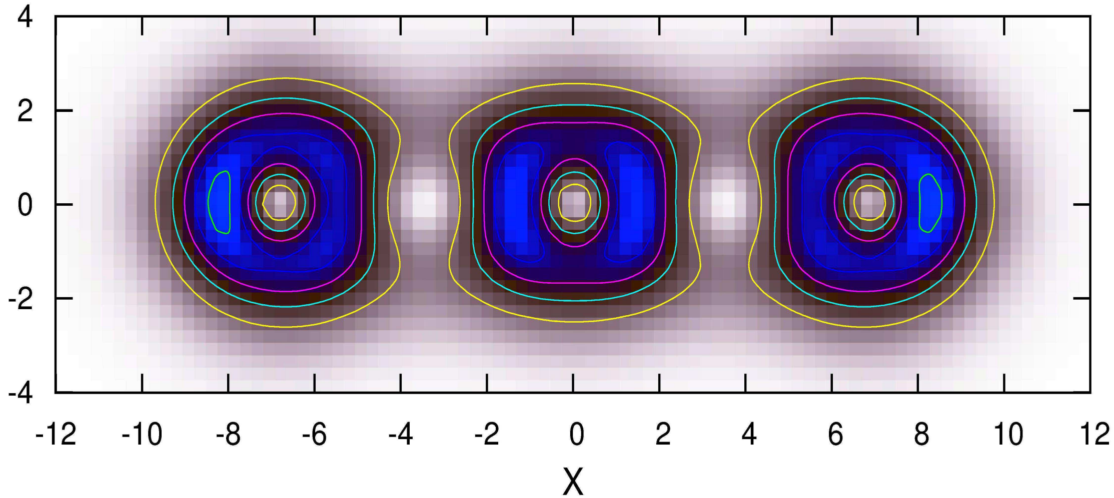
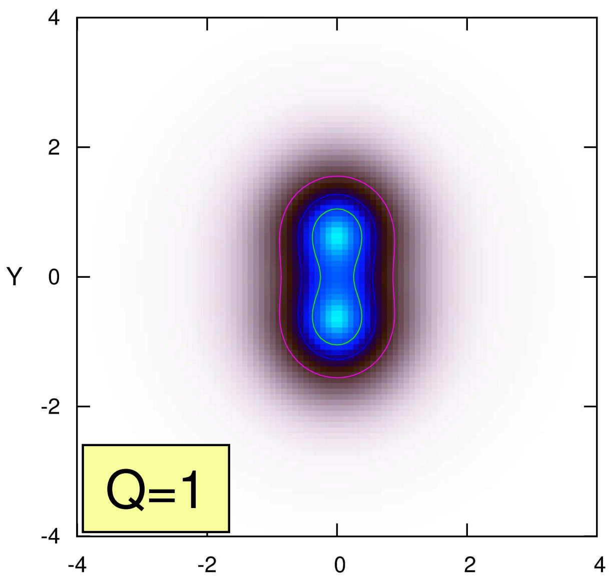
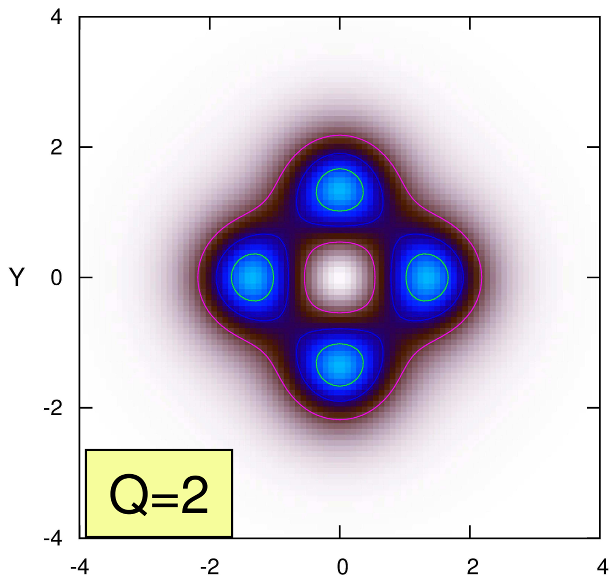
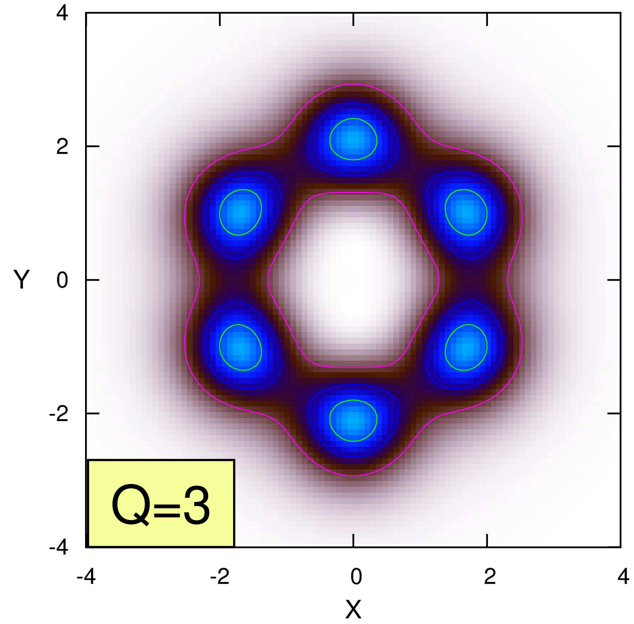


Baby Skyrme model: solitons

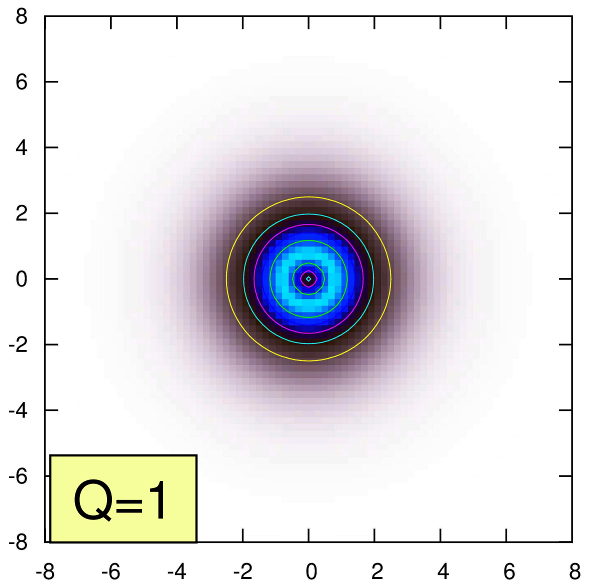
- **Easy plane potential** $U(\phi) = \mu^2 \phi_1^2$

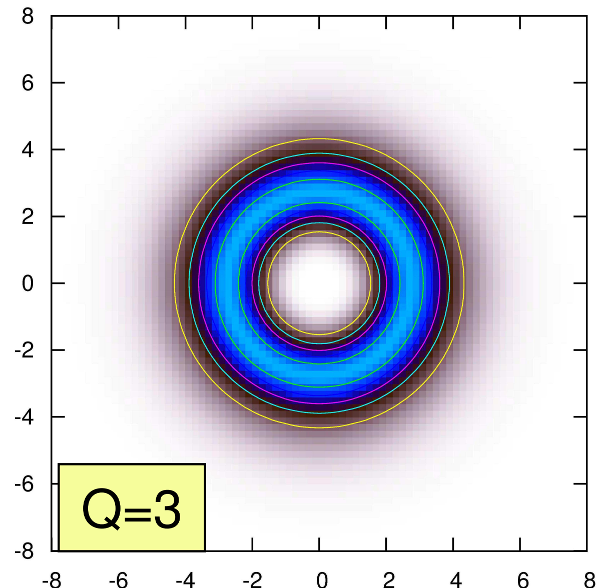


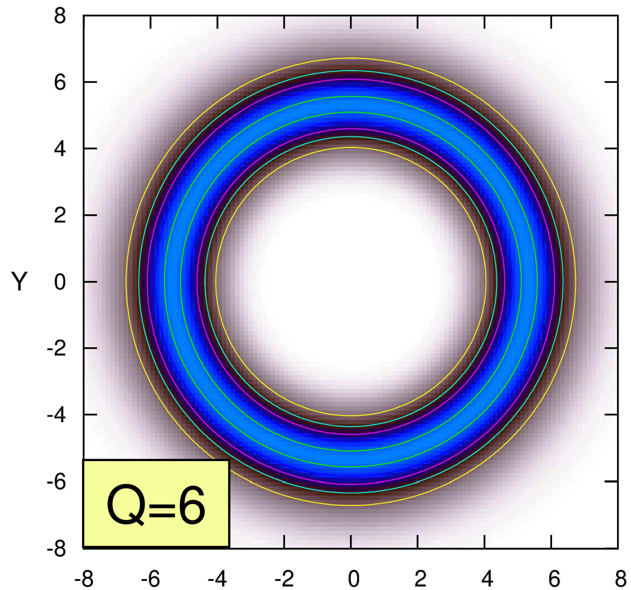




- **Double vacuum potential** $U(\phi) = \mu^2 (1 - \phi_3^2)$

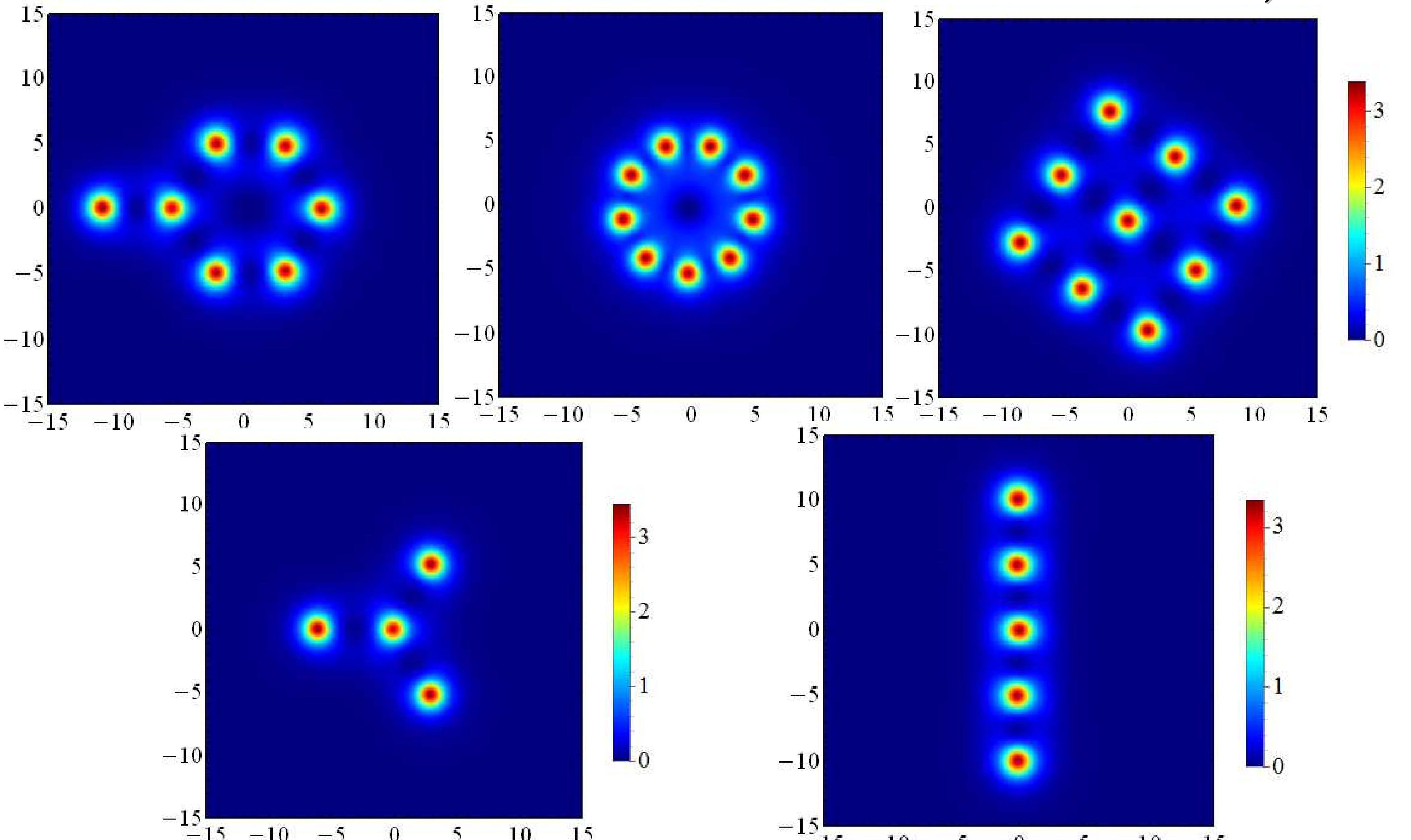


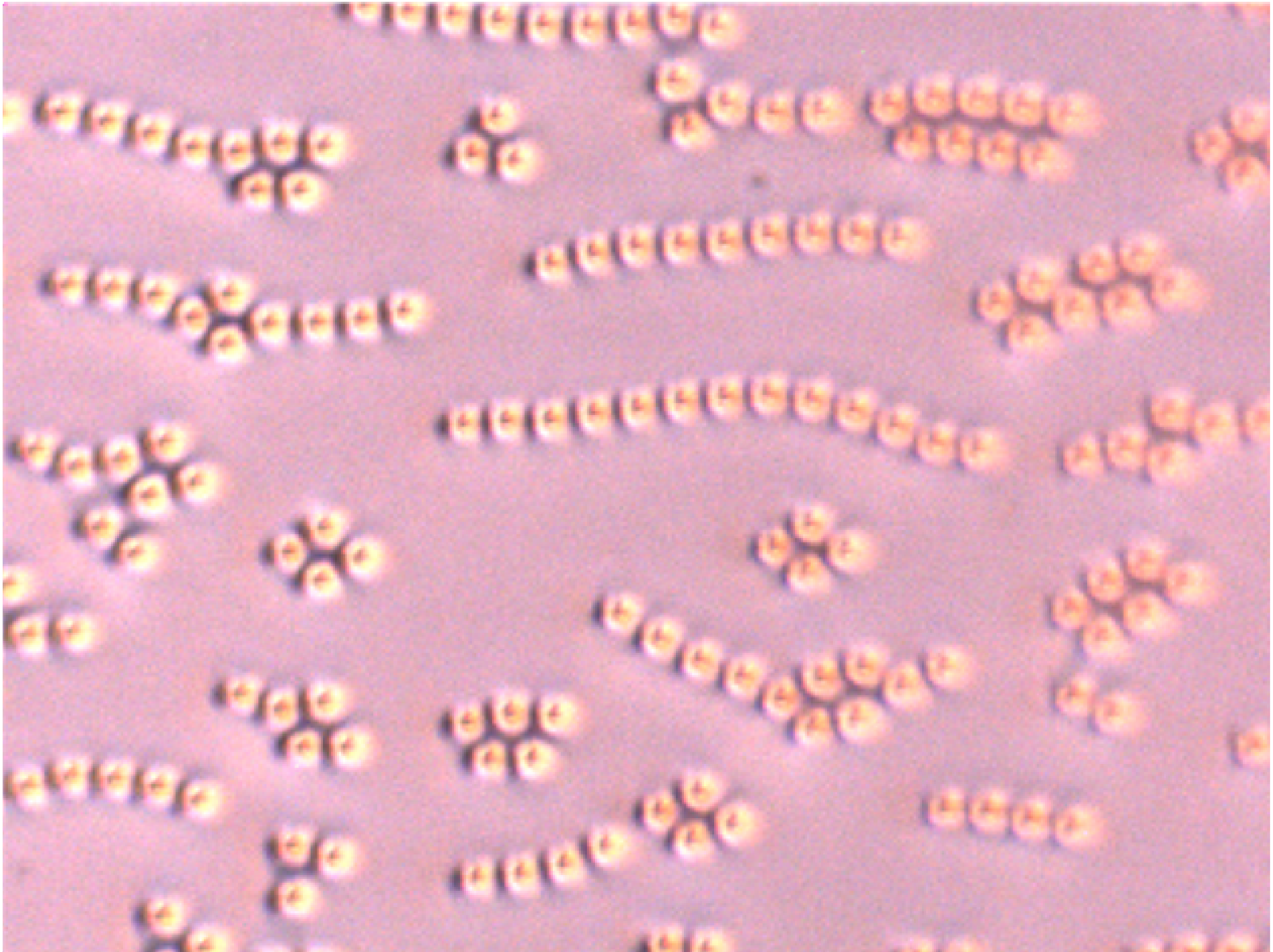




Baby Skyrme model: solitons

- **Weakly bounding potential** $U(\phi) = \mu^2 \left(\alpha(1-\phi_3) + (1-\alpha)(1-\phi_3)^4 \right)$





Gauged baby Skyrme model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \frac{1}{4}\left(D_\mu\vec{\phi}\times D_\nu\vec{\phi}\right)^2 - V(\phi)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \qquad D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + gA_\mu\vec{\phi}\times\phi_\infty$$

$\phi : S^2 \to S^2;\quad \phi_\infty = (0,0,1)$ → SO(2) ≃ U(1) unbroken symmetry group

$$(\phi_1 + i\phi_2) = \phi_\perp \to \phi'_\perp = U\phi_\perp; \quad U = e^{ig\alpha} \qquad A_\mu \to A'_\mu = A_\mu + \frac{i}{g}U\partial_\mu U^{-1}$$

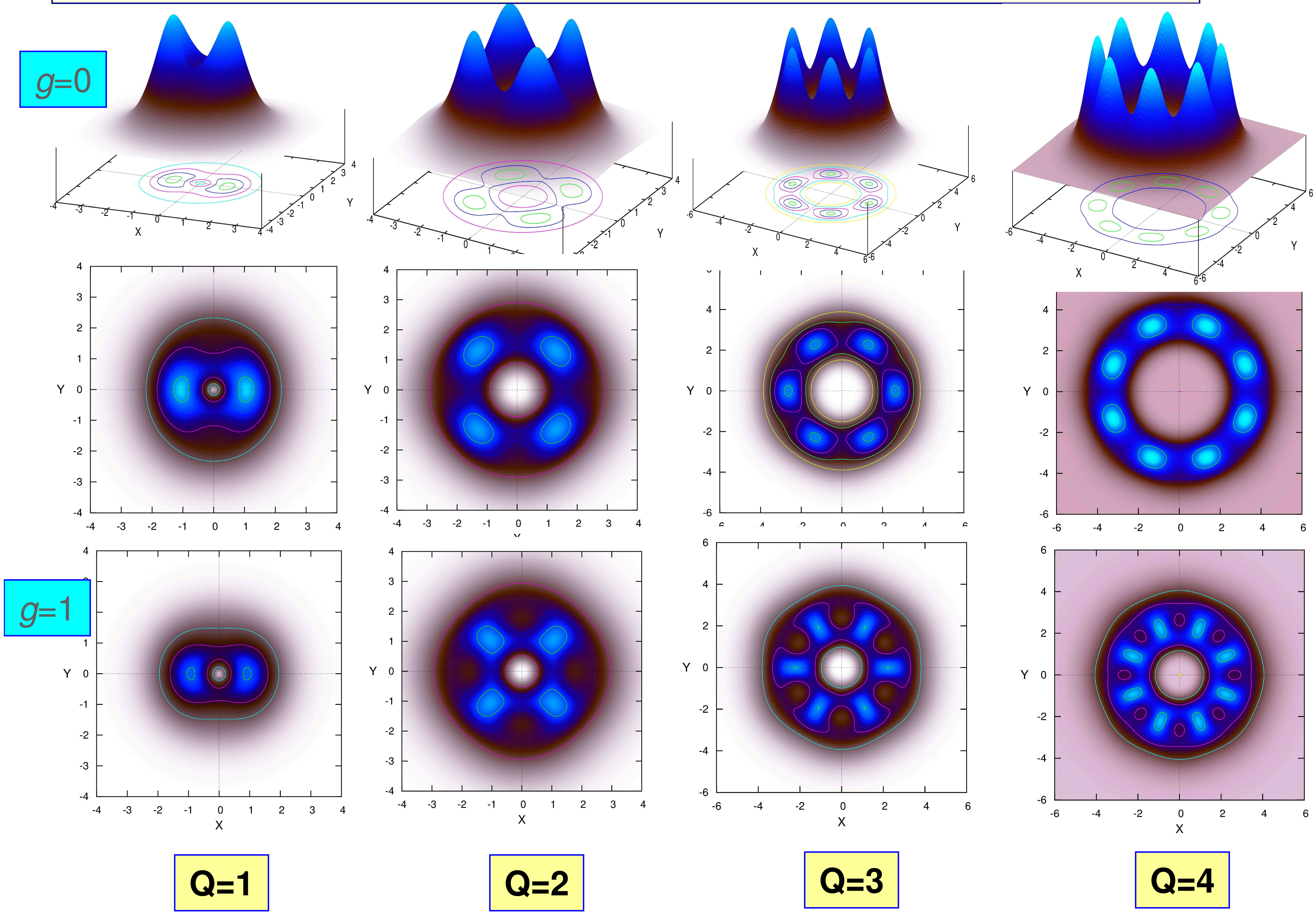
- ***Field equations:***

$$D_\mu\vec{J}^\mu = \frac{V}{\vec{\phi}}\times\vec{\phi}$$

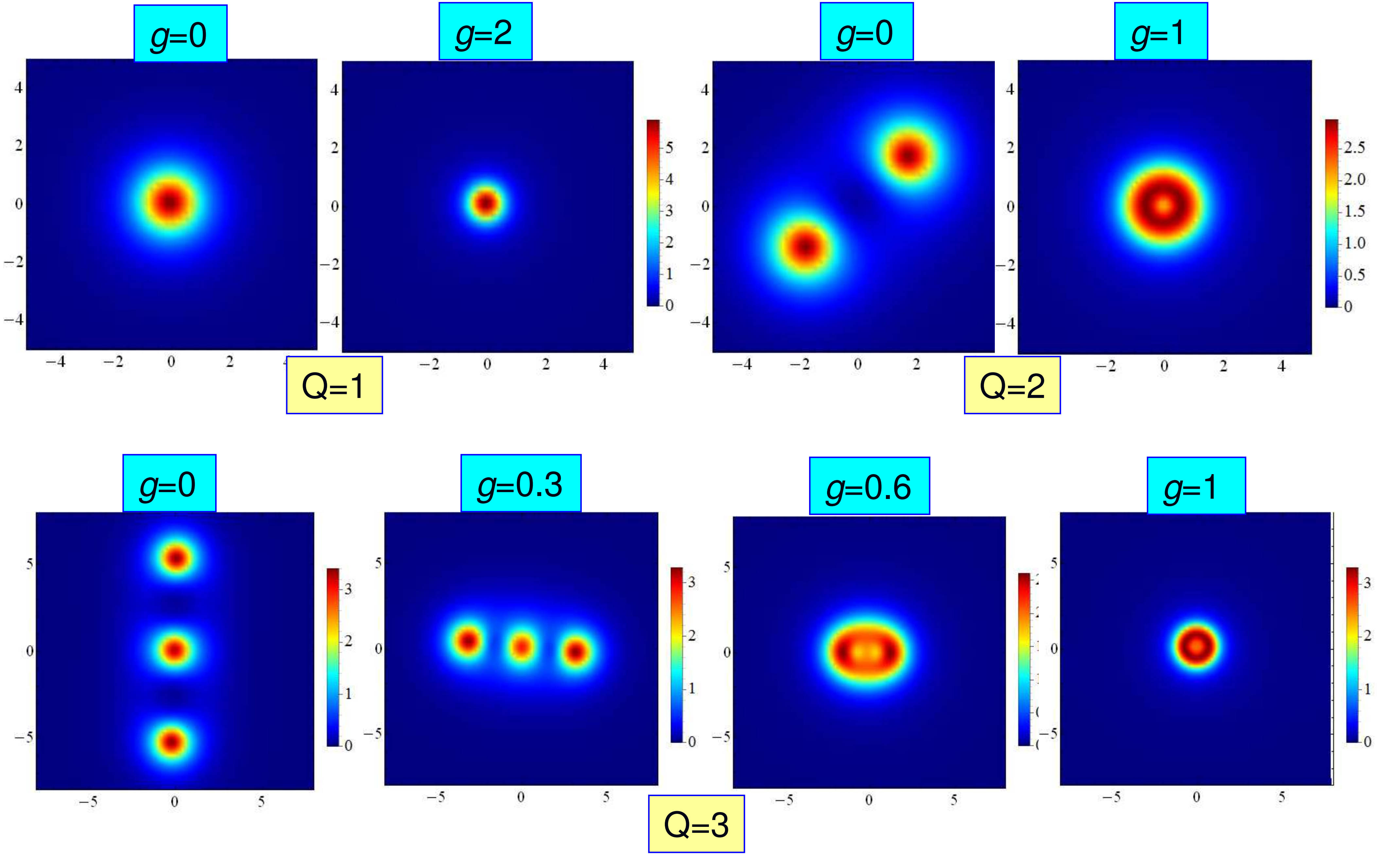
$$\partial_\mu F^{\mu\nu} + \frac{c}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = g\vec{\phi}_\infty\cdot\vec{J}^\nu$$

- ***Current:*** $\vec{J}^\mu = \vec{\phi}\times D^\mu\vec{\phi} - D_\nu\vec{\phi}(D^\nu\vec{\phi}\cdot\vec{\phi}\times D^\mu\vec{\phi})$

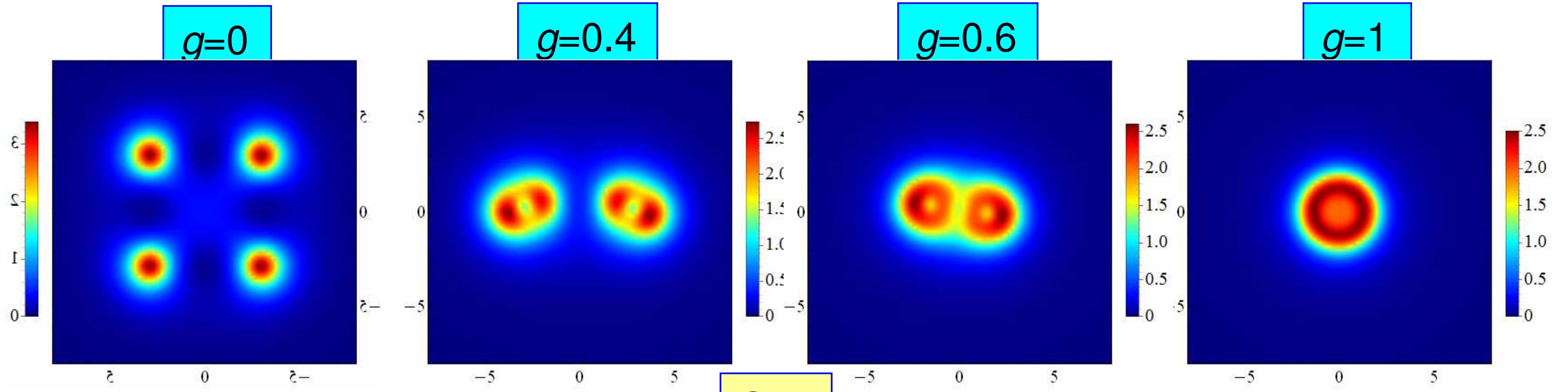
Symmetry breaking Ward potential: $U(\phi) = m^2(1-\phi_3^2)(1-\phi_1^2)$



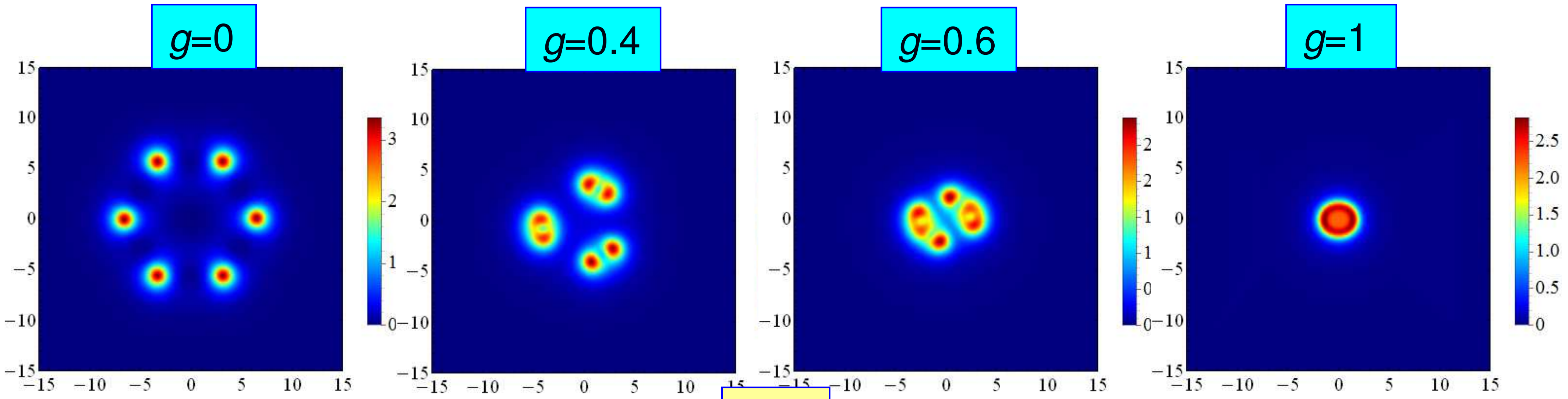
Weakly bounding potential: $U(\phi) = \mu^2 \left[\alpha(1-\phi_3) + (1-\alpha)(1-\phi_3)^4\right]$



Weakly bounding potential: $U(\phi) = \mu^2 \left[\alpha(1-\phi_3) + (1-\alpha)(1-\phi_3)^4\right]$

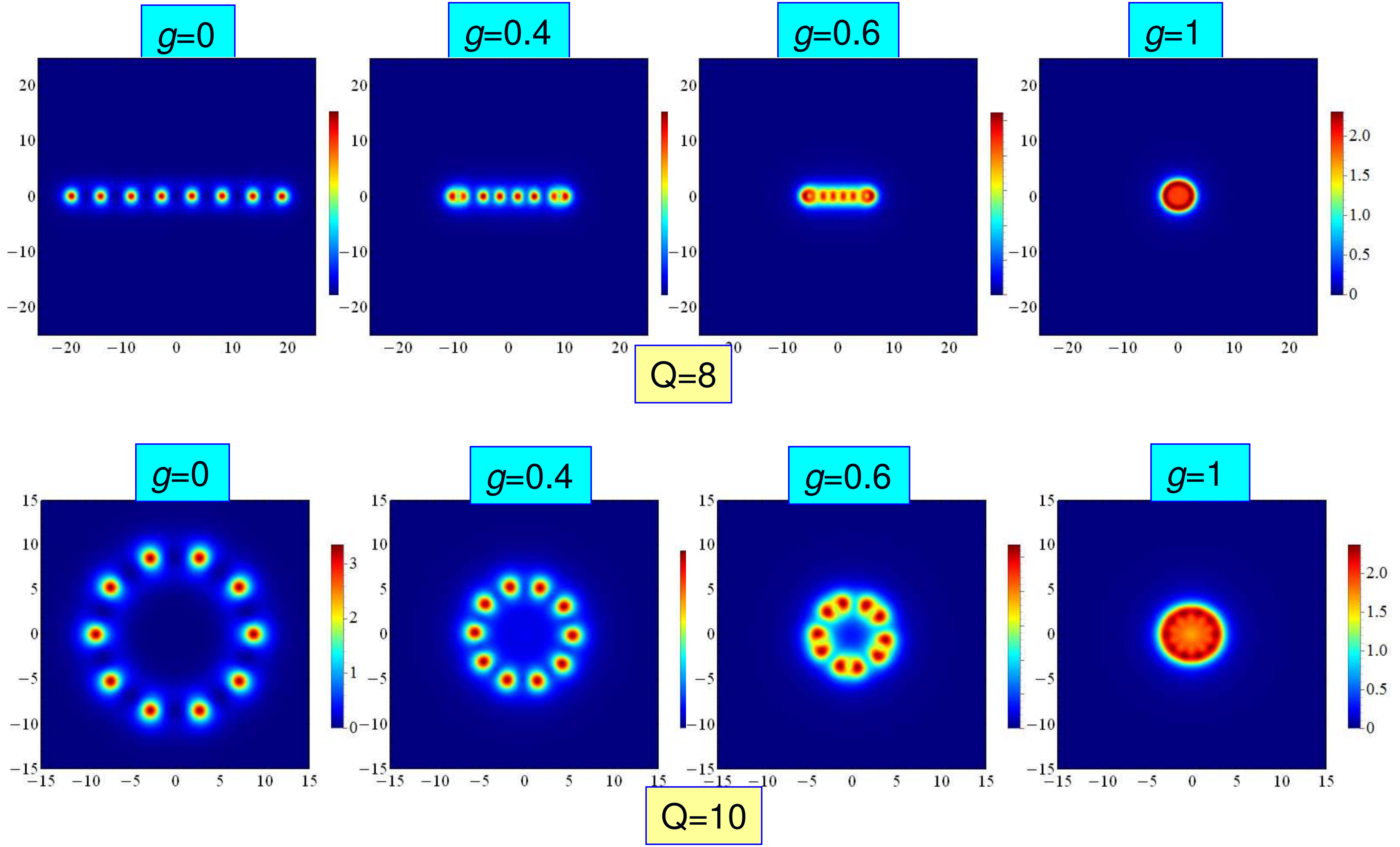


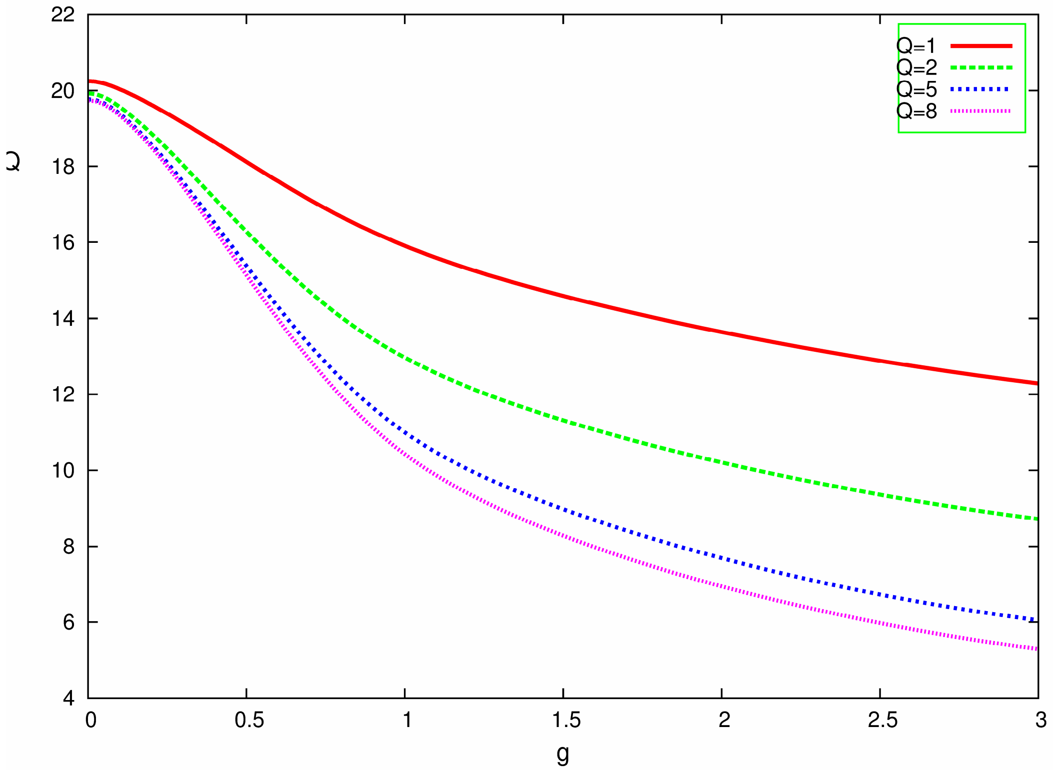
Q=3

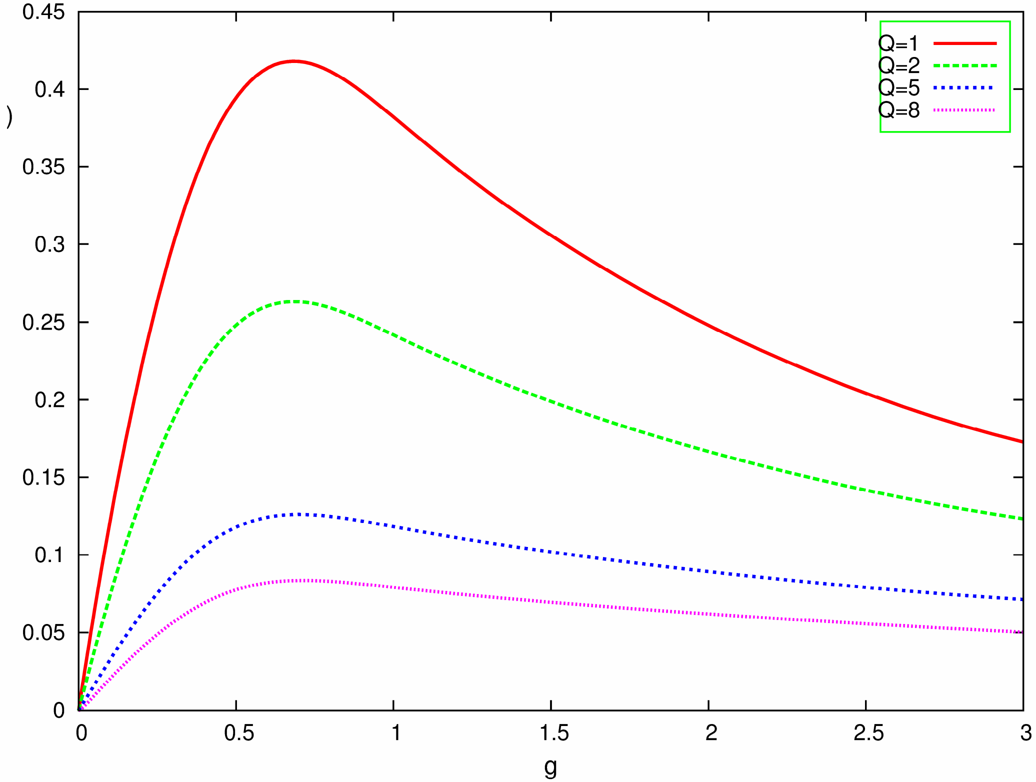


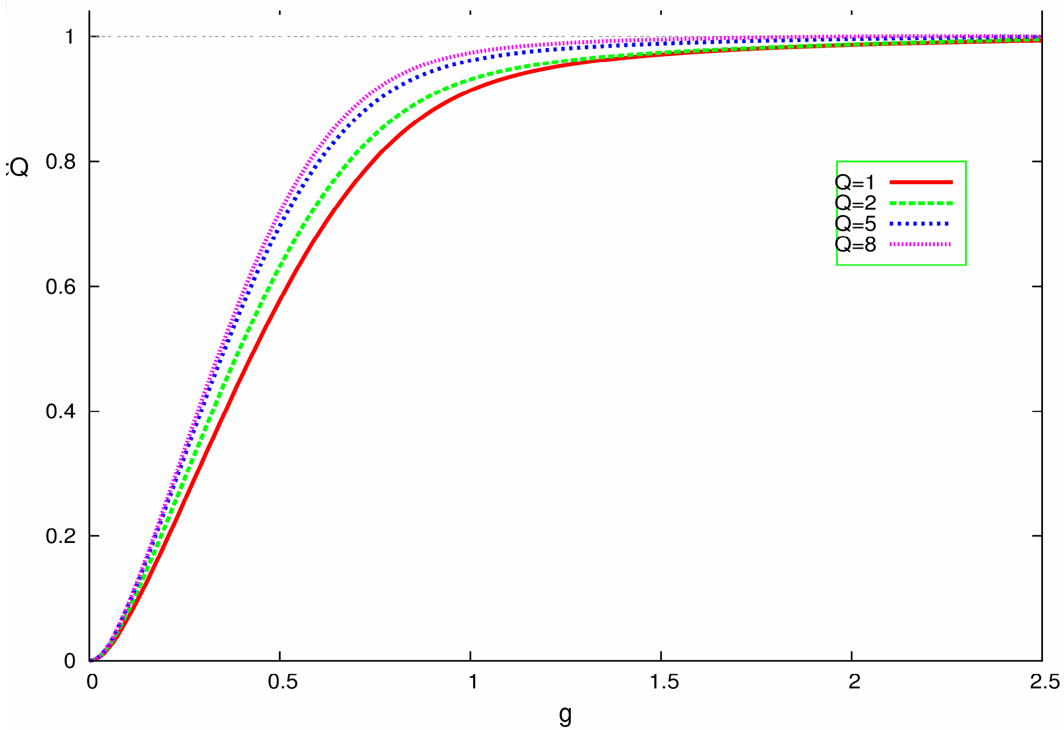
Q=6

Weakly bounding potential: $U(\phi) = \mu^2 \left[\alpha(1-\phi_3) + (1-\alpha)(1-\phi_3)^4\right]$









- ***There is no electric field in the gauged planar Skyrme model***
- ***In the strong coupling limit the total magnetic flux is quantized, $g\Phi=Q$***
- ***The energy of the soliton is decreasing as g grows***

Skyrme model in 3d

- **The Skyrme field:** $U = \phi_0 \mathbb{I} + i\sigma^a \cdot \pi^a$ $\qquad \phi^a = (\phi_0, \pi^a); \quad \phi^a \cdot \phi^a = 1$

$$L = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\partial_\mu \phi^a \partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^b \partial^\nu \phi^b) - m^2(1 - \phi^a \phi^a_\infty)$$

Sigma-model term — Skyrme term — Potential term

- **The topological charge:**

$$B = \frac{1}{12\pi^2} \int d^3x \; \varepsilon_{abcd} \varepsilon^{ijk} \phi^a \partial_i \phi^b \partial_j \phi^c \partial_k \phi^d$$

Topological bound:

$$E \geq 12\pi^2 |B|$$

Topological bound is not saturated in the Skyrme model, solitons are interacting (m=0 → dipole forces)

Skyrme model

- **Spherically symmetric skyrmion:**

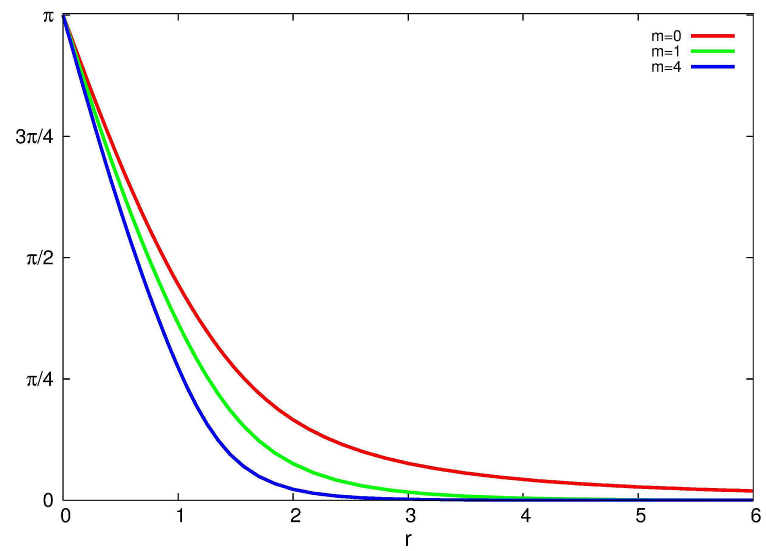
$$U(r) = \exp\left[i\tau^a \hat{r}^a F(r)\right]$$

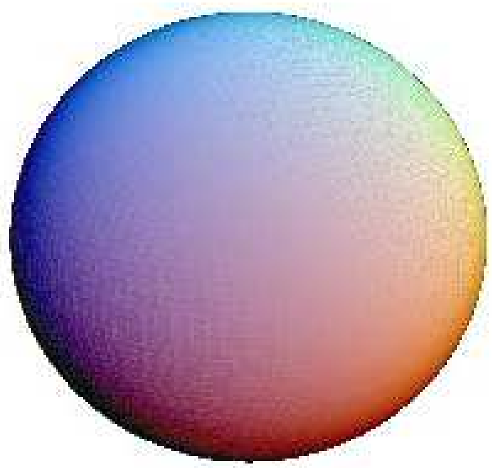
(Hedgehog ansatz)

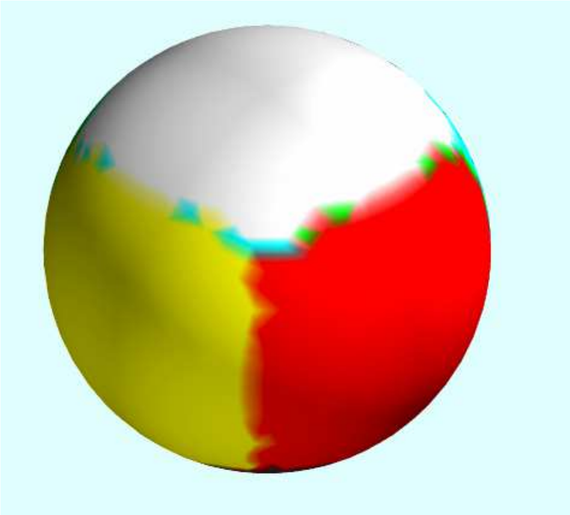
$$Q = \frac{1}{\pi}\left[F(r) - \frac{\sin 2F(r)}{2}\right]_0^\infty$$

The boundary conditions

$$F(0) = \pi,\ F(\infty) = 0 \quad \Rightarrow \quad Q = 1$$







$$U(r) = \sigma + \pi^a \cdot \tau^a = \cos F(r) + i\hat{\mathbf{n}} \cdot \tau \sin F(r)$$

$$\phi^a = (\sigma, \pi^1, \pi^2, \pi^3)$$

$$L = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}(\partial_\mu \phi^a \partial_\mu \phi^a)^2 + \frac{1}{2}(\partial_\mu \phi^a \partial_\nu \phi^a)(\partial^\mu \phi^b \partial^\nu \phi^b) - m^2(1-\sigma)$$

Skyrmions from instantons

M.Atiyah and N.Manton, Phys. Lett. B 222, 438 (1989)

SU(2) Yang-Mills

$$L = \frac{1}{2g^2}\mathrm{Tr}\ F_{\mu\nu}^2$$

Skyrme model

$$L = \frac{f_\pi^2}{4}\mathrm{Tr}\,(\partial_\mu U \partial^\mu U) + \dots$$

Instanton's holonomy:

$$U(\mathbf{x}) = \mathcal{P}\exp\left(i\int_{-\infty}^{\infty} dx_0 A_0(\mathbf{x}, x_0)\right) \in SU(2) \xrightarrow[\mathbf{x}\to\infty]{} \mathbb{I}$$

B=1

- **YM Instanton**

$$A_0 = i\hat{r^a}\cdot\tau^a\left(\frac{1}{r^2+x_0^2+\lambda} - \frac{1}{r^2+x_0^2}\right)$$

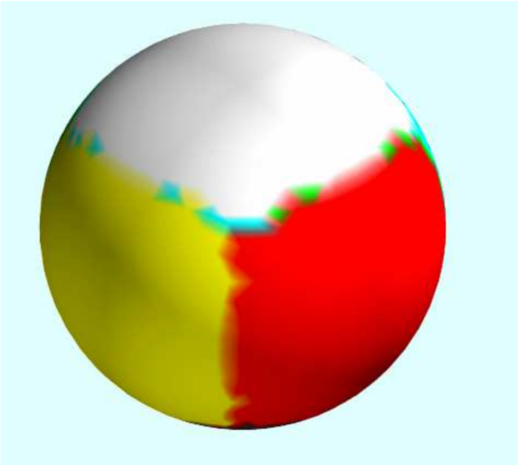
- **Skyrmion**

$$\begin{cases} U(r) = \exp\left[i\tau^a\hat{r}^a F(r)\right] \\ F(r) = \pi\left(1 - \dfrac{r}{\sqrt{r^2+\lambda^2}}\right) \end{cases}$$

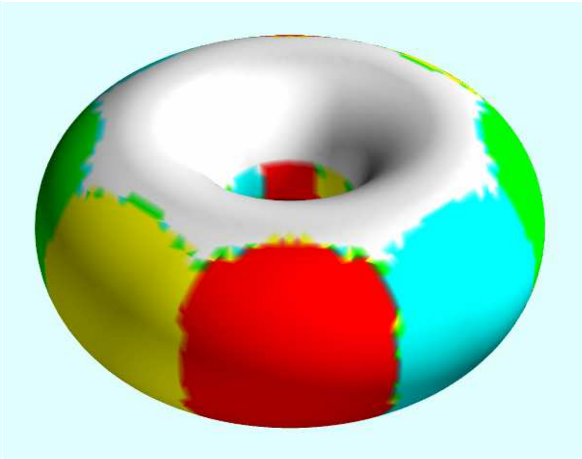
Pontryagin index → **Skyrmion winding number**

$$\frac{1}{16\pi^2}\int d^4x F_{\mu\nu}\widetilde{F}_{\mu\nu} \quad\Rightarrow\quad \frac{1}{24\pi^2}\varepsilon_{ijk}\int d^3x \mathrm{Tr}\ \left[(U^\dagger\partial^i U)(U^\dagger\partial^j U)(U^\dagger\partial^k U)\right]$$

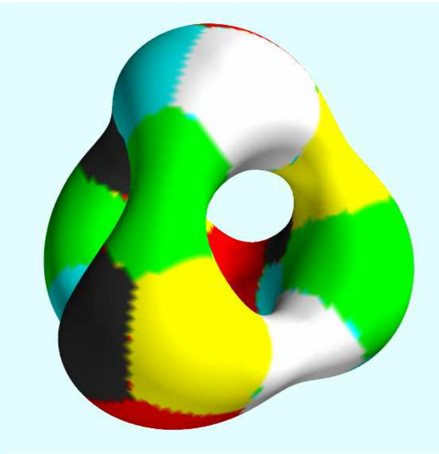
Skyrmions



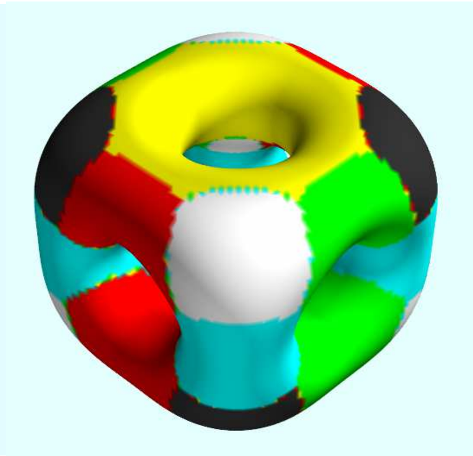
Q=1



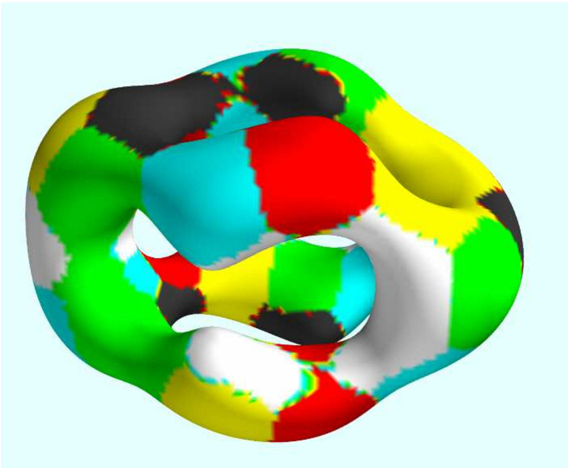
Q=2



Q=3



Q=4



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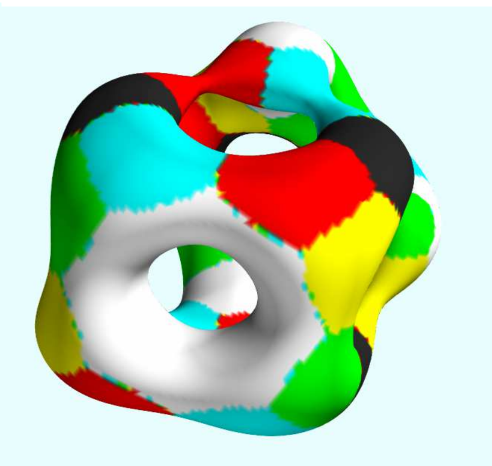

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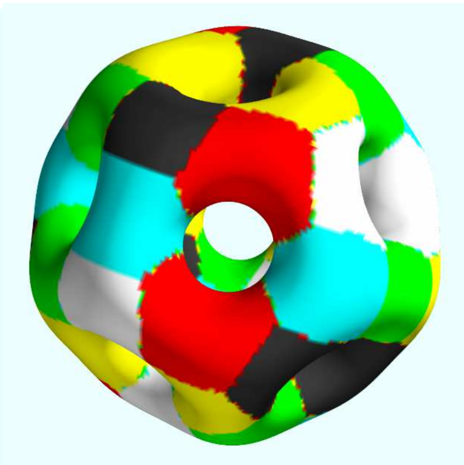
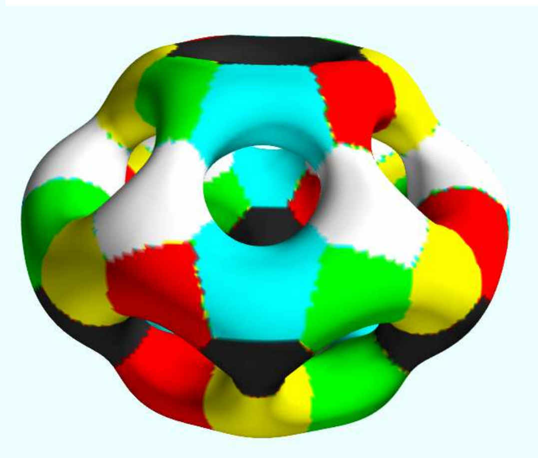


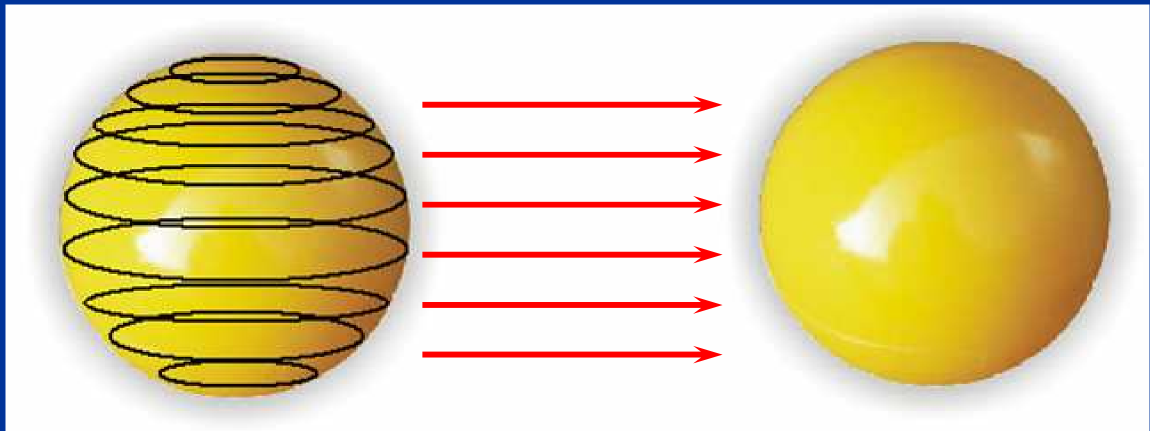
Q=7



Q=8

Rational map Skyrmions

The Skyrme field is effectively a map *U*: $S_3 \rightarrow SU(2) \sim S_3$



(N.S. Manton, C.Houghton & P.Sutcliffe, 1998)

Stereographic Projection $z = \tan(\theta/2)e^{i\varphi}$

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}\left(\frac{z+\bar{z}}{1+z\bar{z}}, i\frac{z^*-z}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+\bar{z}}\right)$$
$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Domain space

The idea of the rational map ansatz:

- Separate the radial and the angular dependence of the Skyrme field **as**

$$U = \exp\{if(r)\hat{\mathbf{n}}_Z \cdot \sigma\}$$

- *I*dentify spheres S_2 with concentric spheres in compactified R_3
- *I*dentify target space S_2 with spheres of latitude on S_3

$$\hat{n}_Z = \left(\frac{Z+\bar{Z}}{1+Z\bar{Z}}, i\frac{\bar{Z}-Z}{1+Z\bar{Z}}, \frac{1-Z\bar{Z}}{1+Z\bar{Z}}\right)$$

$$Z = P(z)/Q(z)$$

Target space

Rational map approximation

- **Static energy:** $E = 4\pi \int \left(r^2 f'^2 + 2Q(f'^2+1)\sin^2 f + W \frac{\sin^4 f}{r} \right) dr$

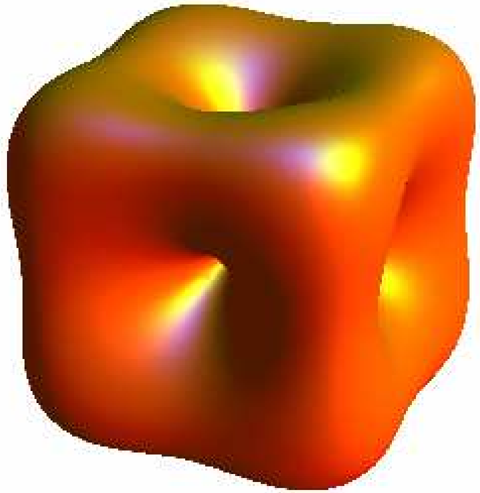
$$4\pi Q = \int \left(\frac{1+|z|^2}{1+|Z|^2} \left| \frac{dZ}{dz} \right| \right)^2 \frac{dz d\bar{z}}{(1+|z|^2)^2}$$

$$W = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|Z|^2} \left| \frac{dZ}{dz} \right| \right)^4 \frac{dz d\bar{z}}{(1+|z|^2)^2}$$

The holomorphic maps of degree Q:

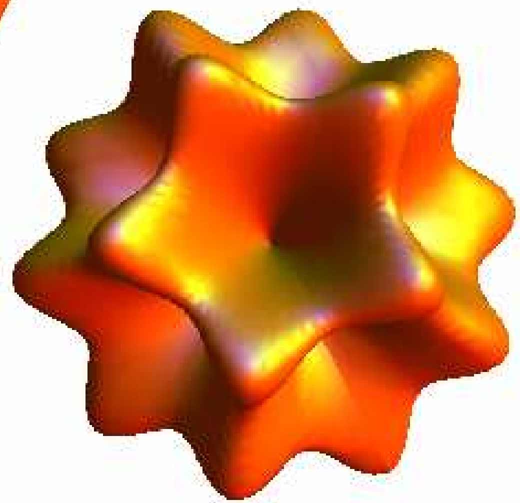
Q= 4: $Z(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2i\sqrt{3}z^2 + 1}$

(Octahedral Skyrmions)



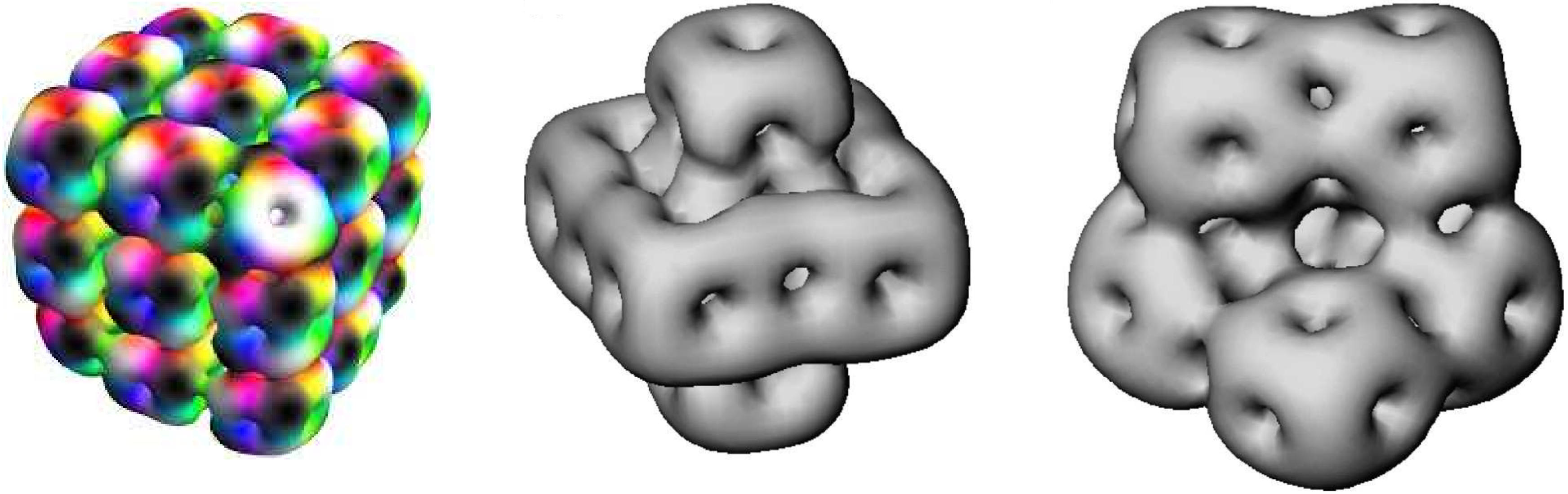
Q= 7: $Z(z) = \frac{z^7 - 7z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1}$

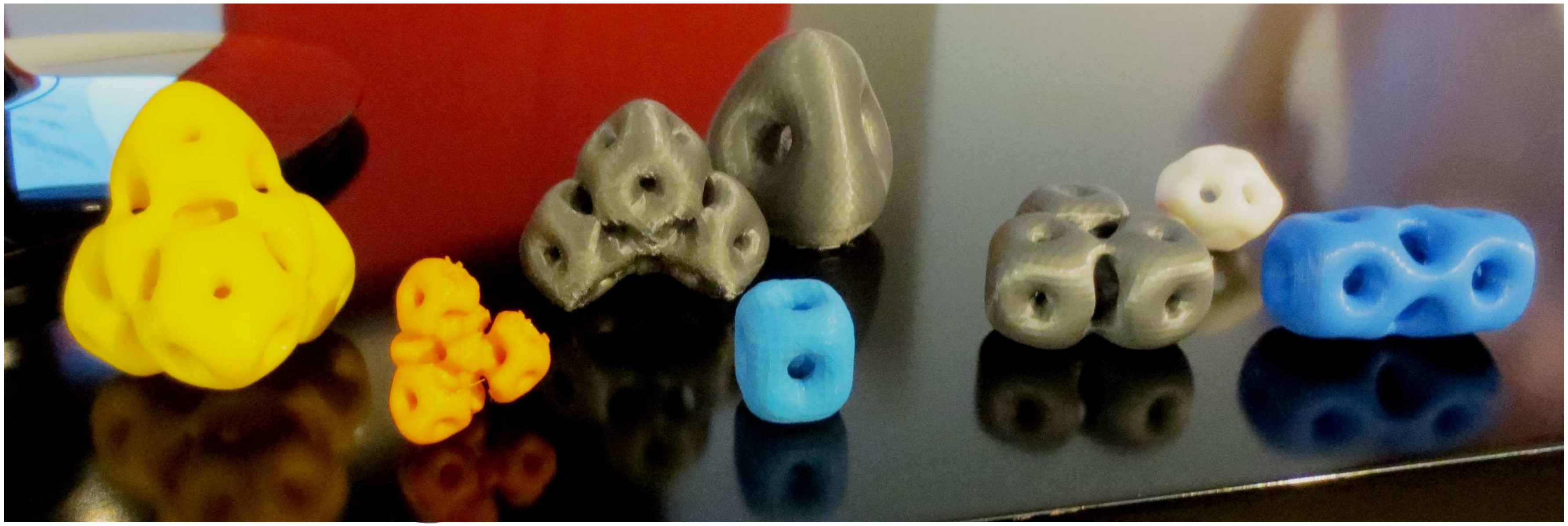
(Icosahedral Skyrmions)



Skyrmions

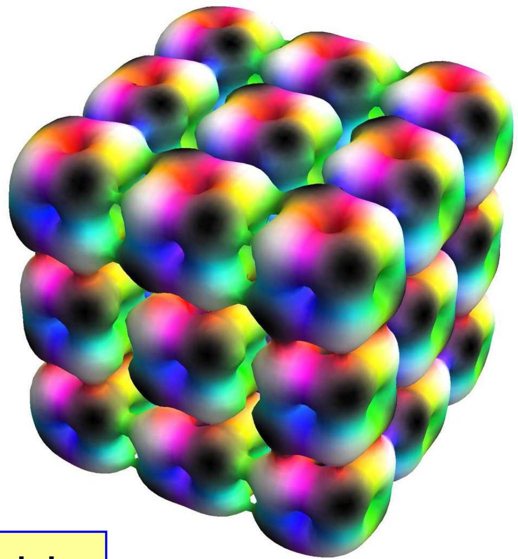
(N.Manton et al)



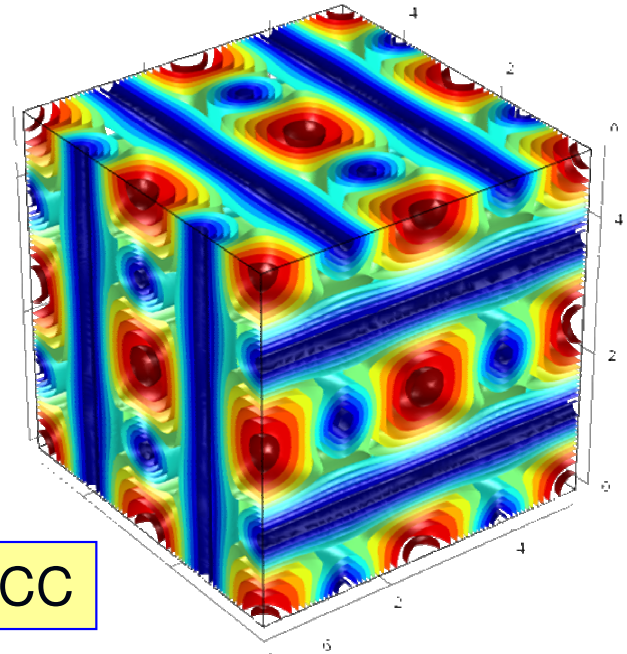


Skyrme crystals

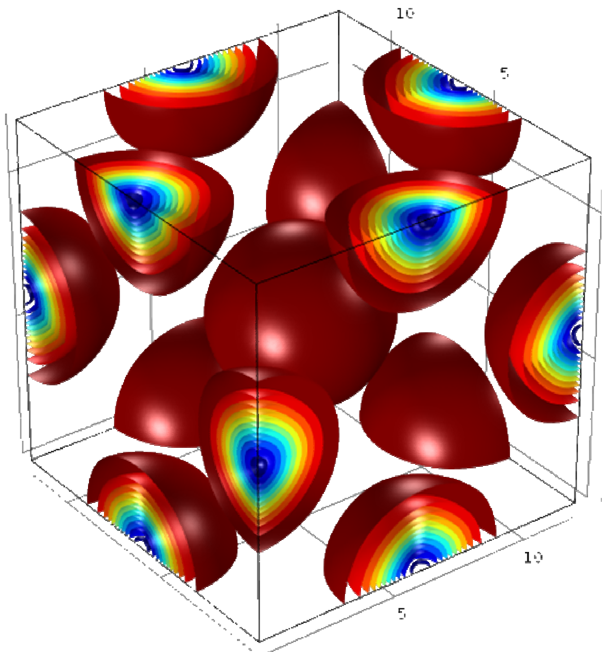
(Klebanov, Kugler, Shtrikman, Manton..)



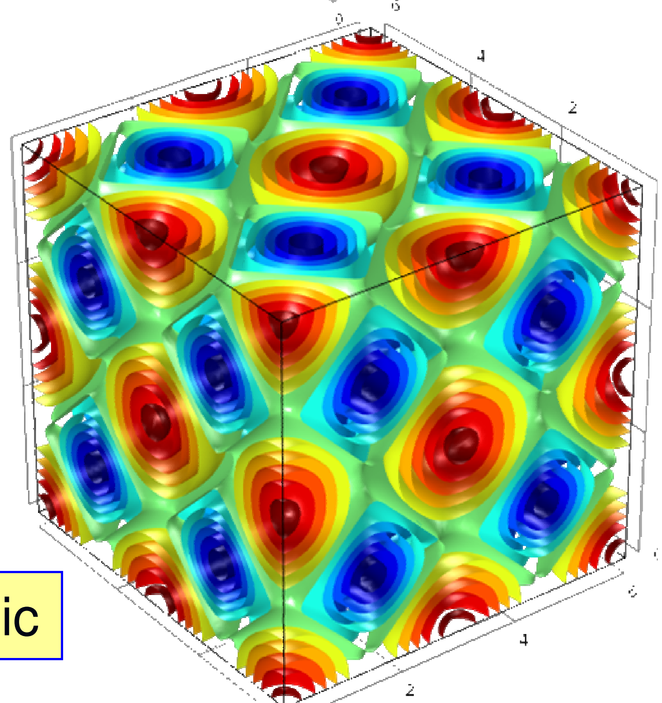
Simple cubic



BCC



FCC



½ Simple cubic

U(1) gauged Skyrme model

B.Piette & D.H.Tchrakian (1997)
E.Radu & D.H.Tchrakian (2005)

$$L = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathrm{Tr}\, \left(D_\mu U\, D^\mu U^\dagger\right) + \frac{1}{16} \mathrm{Tr}\, \left(\left[D_\mu U\, U^\dagger,\, D_\nu U\, U^\dagger\right]^2\right) + m^2 \mathrm{Tr}\, (U - \mathbb{I})$$

$$L = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^a D^\mu \phi^a - \frac{1}{2} (D_\mu \phi^a D^\mu \phi^a)^2 + \frac{1}{2} D_\mu \phi^a D_\nu \phi^a)(D^\mu \phi^b D^\nu \phi^b) - 2\, m^2\, (1 - \phi_0)$$

$$D_\mu \phi_\alpha = \partial_\mu \phi_\alpha - g\, A_\mu \varepsilon_{\alpha\beta}\, \phi_\beta \qquad D_\mu \phi_A = \partial_\mu \phi_A, \qquad \alpha,\, \beta = 1,\, 2,\; A = 0,\, 3$$

- **U(1) gauge symmetry:**

$$U \to e^{i\, g\, \frac{\alpha}{2} \tau_3}\, U\, e^{-i\, g\, \frac{\alpha}{2} \tau_3}, \quad \text{or} \quad \phi_1 + i\, \phi_2 \to e^{-i\, g\, \alpha}\, (\phi_1 + i\, \phi_2)\,, \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

- **Vacuum asymptotic expansion** $\delta U \sim (1 - v_0)\mathbb{I} + i v_k\, \tau_k, \qquad A_\mu = a_\mu + V\, \delta_{0\mu}$

Linearized equations for parturbations $m^{\pm}_{eff} = \sqrt{m^2 - g^2\, V^2}, \qquad m^{(v_3)}_{eff} = m$

$$\partial_i^2 v_a - \left[m^2\, v_a - g^2\, V^2\, (v_1\, \delta_{a1} + v_2\, \delta_{a2})\right] = 0 \quad \Longrightarrow \quad |\, g\, V\, | \le m \qquad \underline{A_0(\infty) = V}$$

Potential term is needed to stabilize gauged Skyrmions

U(1) gauged Skyrme model

- **Topological charge** $$B=-\frac{1}{24\pi^2}\int d^3x\,\varepsilon_{ijk}\,\mathrm{Tr}\,\left(\partial_i U U^{-1}\,\partial_j U U^{-1}\,\partial_k U U^{-1}\right)$$

vs

$$B_g=-\frac{1}{24\pi^2}\int d^3x\,\varepsilon_{ijk}\,\mathrm{Tr}\,\left(D_i U U^{-1}\,D_j U U^{-1}\,D_k U U^{-1}\right)$$

Gauge covariant charge

$$Q=B_g-B_{\mathrm{mag}}=B+\int d^3x\,\partial_i\Lambda_i,\qquad B_{\mathrm{mag}}=\frac{ig}{32\pi^2}\int d^3x\,(\varepsilon_{ijk}F_{jk})\,\mathrm{Tr}\left(\{\tau_3,\,\partial_i U\}\,U^{-1}\right)$$

$$\Lambda_i=-\frac{ig}{16\pi^2}\varepsilon_{ijk}\,A_j\,\mathrm{Tr}\,\left(\{\tau_3,\,\partial_k U\}\,U^{-1}\right)\quad\Rightarrow\quad Q=B$$

- **The stress-energy tensor** $T^{\mu\nu}=T^{\mu\nu}_{(M)}+T^{\mu\nu}_{(S)}$ $$T^{\mu\nu}_{(M)}=-2\,F^{\mu\sigma}\,F^{\nu}{}_{\sigma}+\frac{\eta^{\mu\nu}}{2}\,F_{\alpha\beta}\,F^{\alpha\beta}$$

$$T^{\mu\nu}_{(S)}=2\left[D^{\mu}\phi_a D^{\nu}\phi^a-\left(D^{[\mu}\phi^a D^{\alpha]}\phi^b\right)\left(D^{[\nu}\phi_a D_{\alpha]}\phi_b\right)\right]$$
$$-\eta^{\mu\nu}\left((D_\alpha\phi_a)^2-\frac{1}{2}\left(D_{[\alpha}\phi_a\,D_{\beta]}\phi_b\right)^2-2m^2\,(1-\phi_0)\right)$$

- **Electric charge** $$Q_e=\int d^3x\,\partial_i^2 A_0=\oint d\vec{S}\cdot\nabla A_0$$

B=1 gauged Skyrmion

$$U = \phi_0 \mathbb{I} + i\sigma^a \cdot \pi^a$$

Axially symmetric parametrization:

$$\pi_1 = \psi_1(r,\theta)\cos(n\varphi); \quad \pi_2 = \psi_1(r,\theta)\sin(n\varphi); \quad \pi_3 = \psi_2(r,\theta); \quad \phi_0 = \psi_3(r,\theta)$$

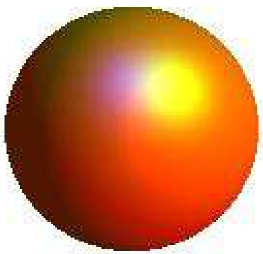
$$A \equiv A_\mu dx^\mu = A_\varphi(r,\theta)d\varphi + A_0(r,\theta)dt$$

$$\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$$

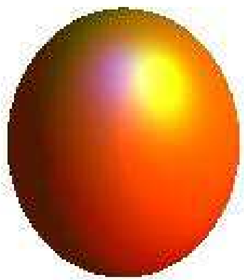
- **Angular momentum:**

$$J = \int d^3x\, T_0^\varphi = -\oint_\infty d\vec{S}\cdot\vec{\nabla}A_0\left(\frac{n}{g} + A_\varphi\right) = -\frac{n}{g}\oint_\infty d\vec{S}\cdot\vec{\nabla}A_0 = n\frac{Q_e}{g}$$

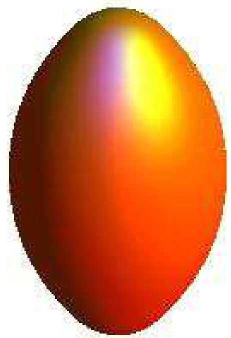
V=0.1



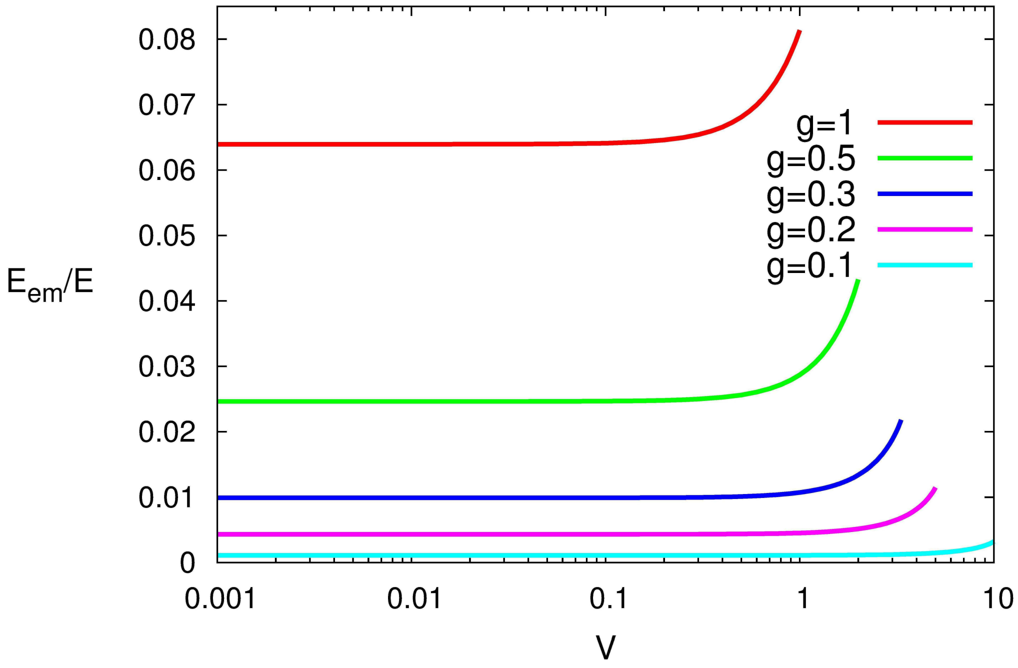
g=0.1

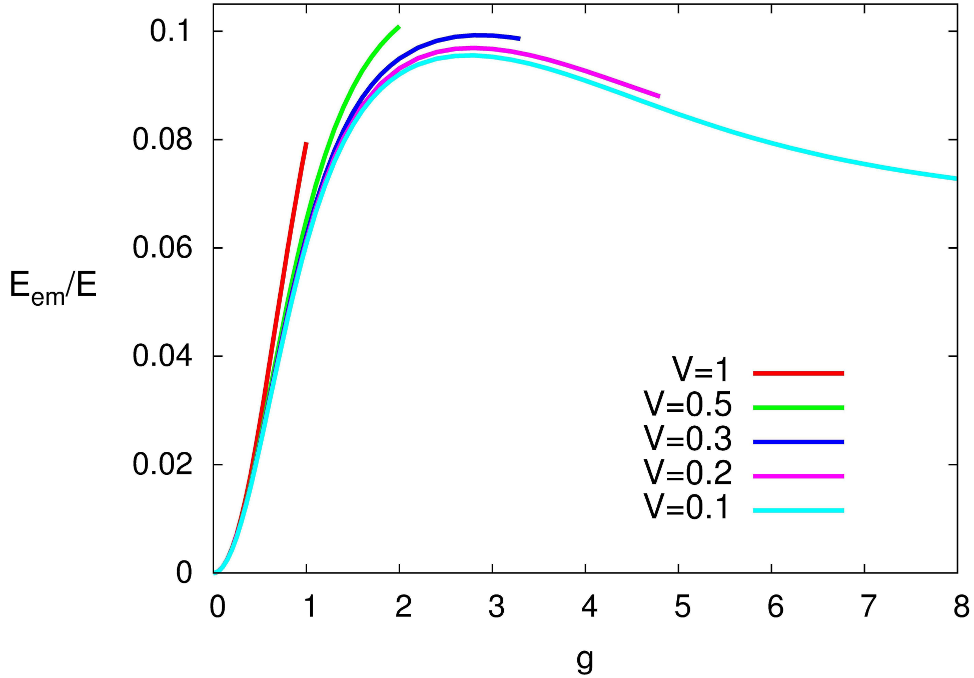


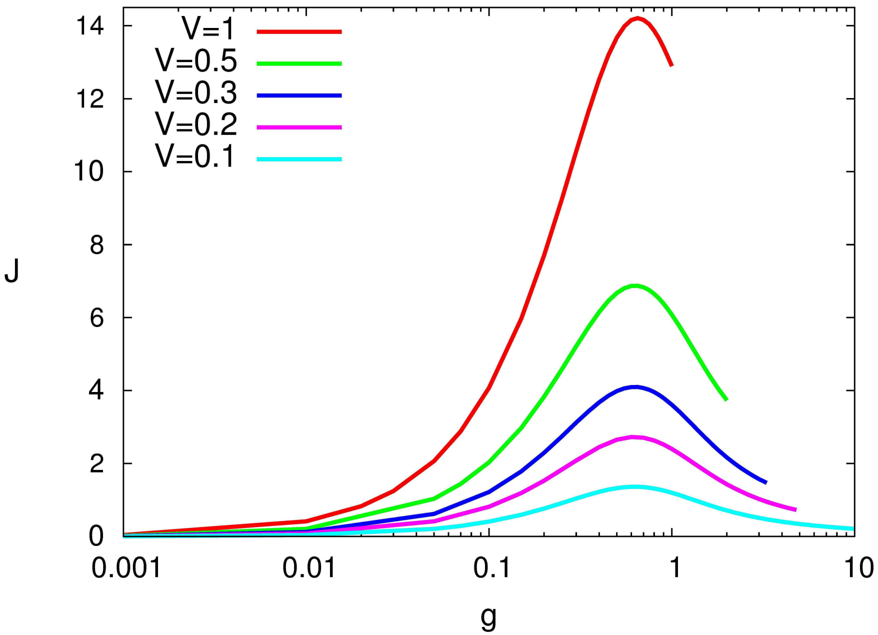
g=0.5

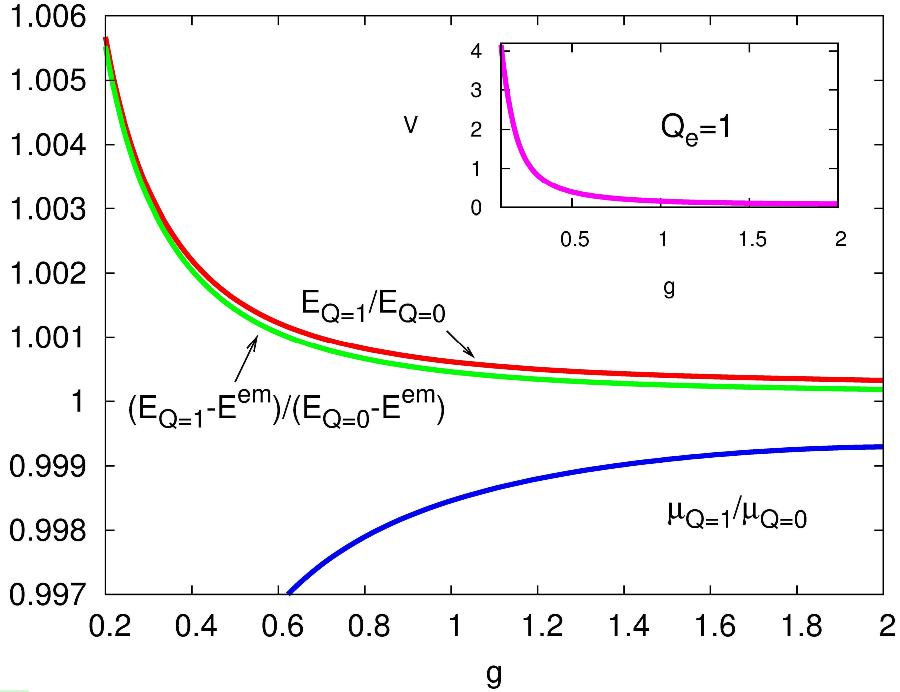


g=2



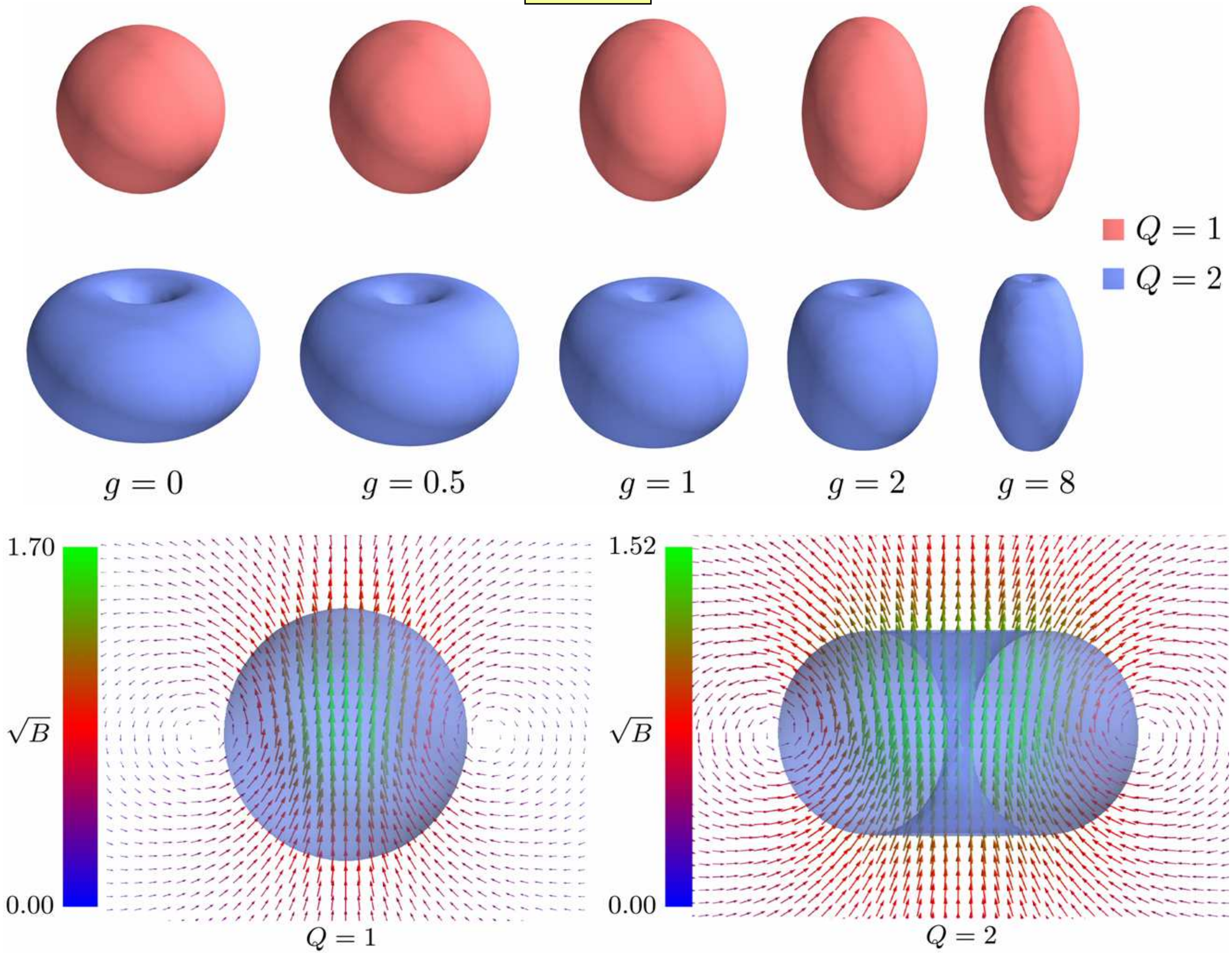




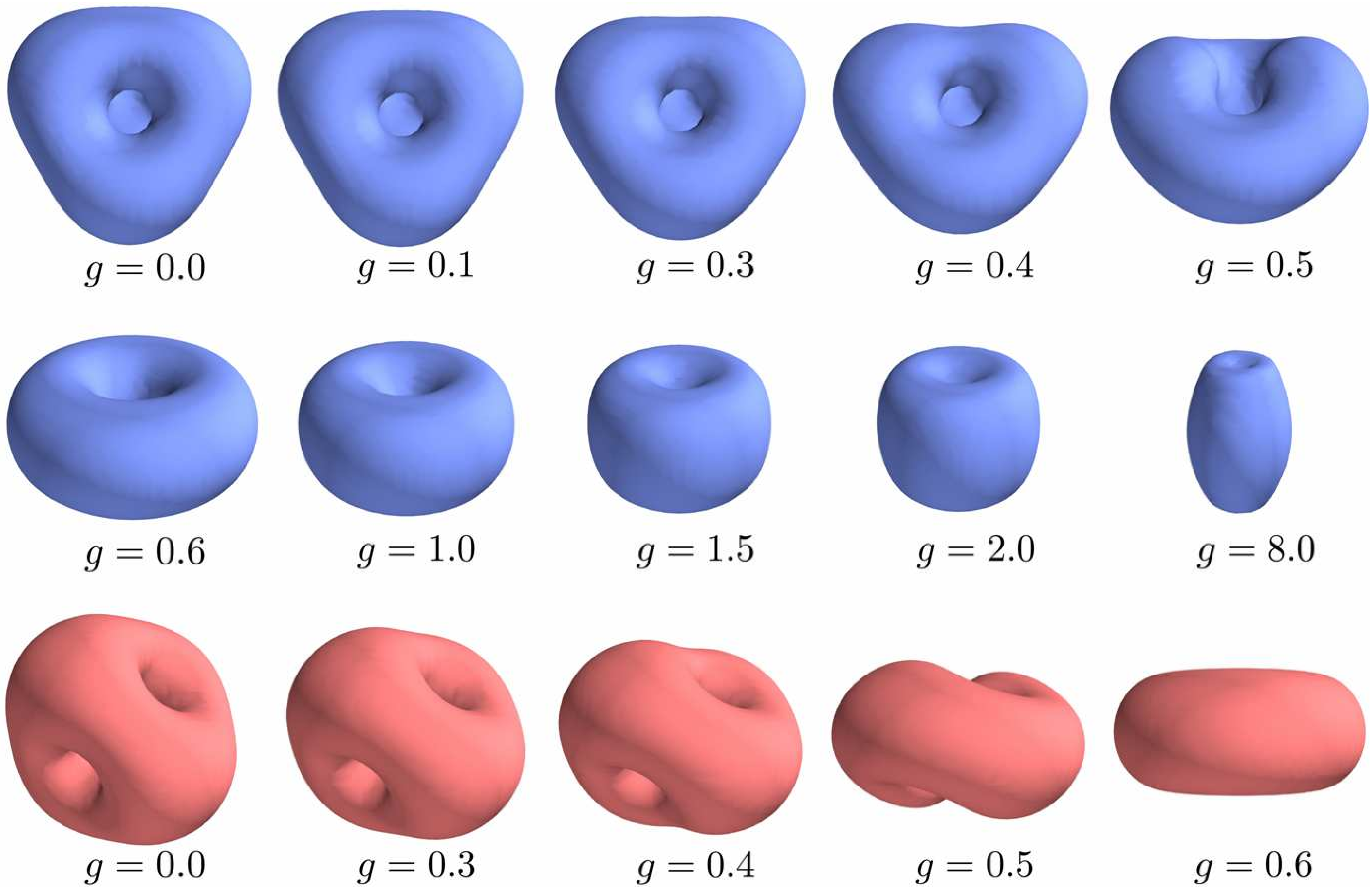


m_n = 939,556 MeV, m_p = 938,272 MeV

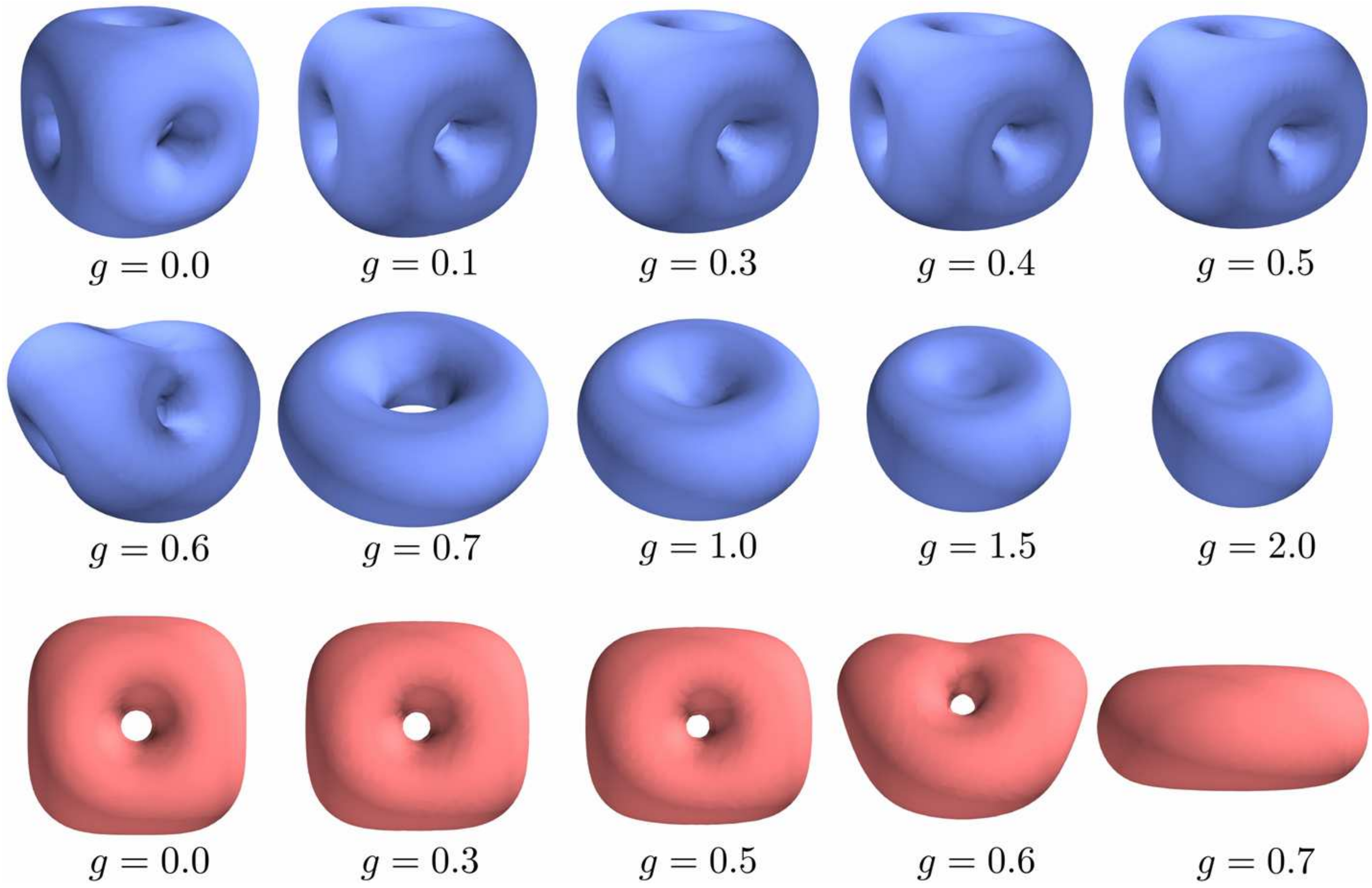
Q=1,2



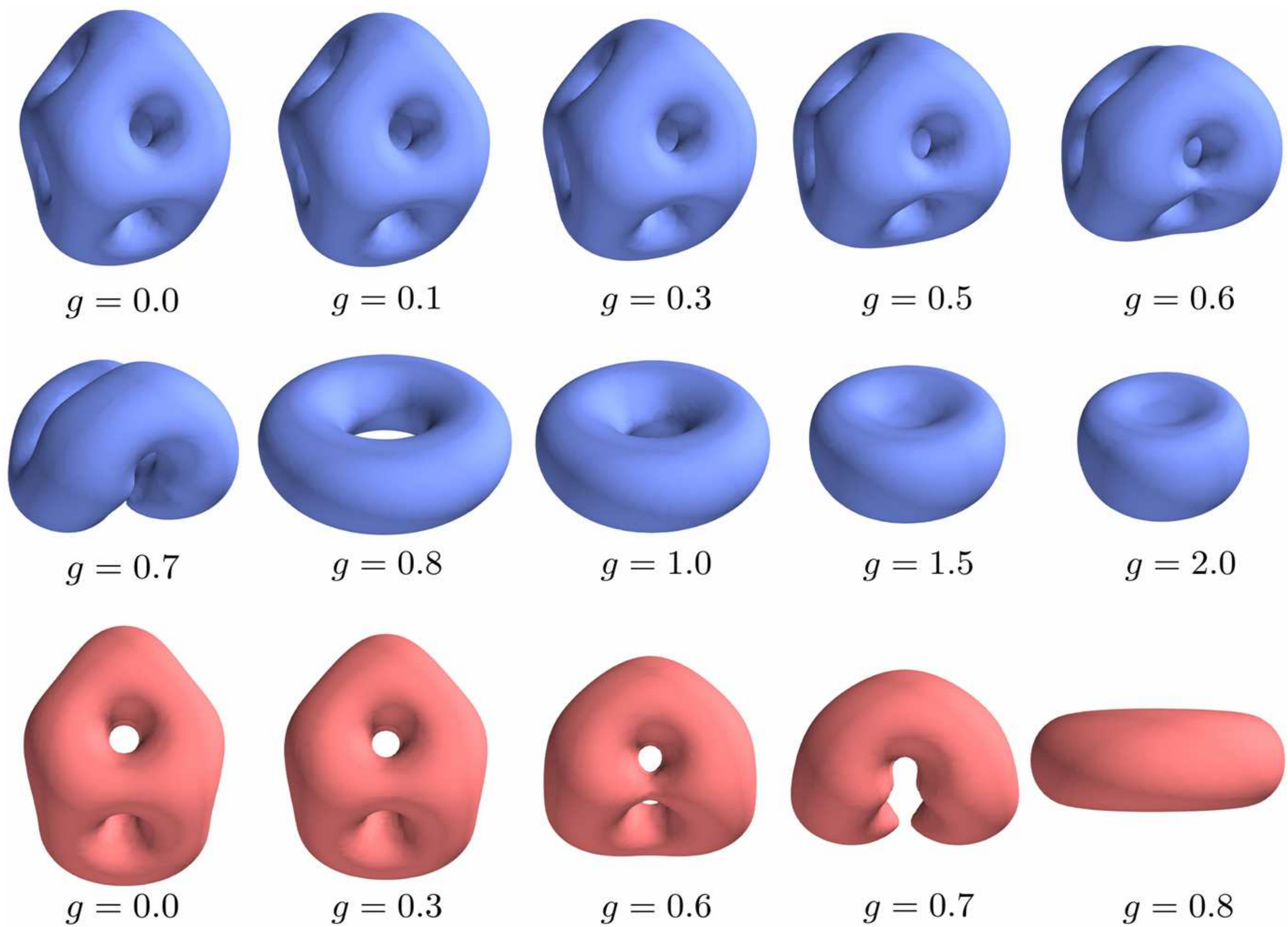
Q=3

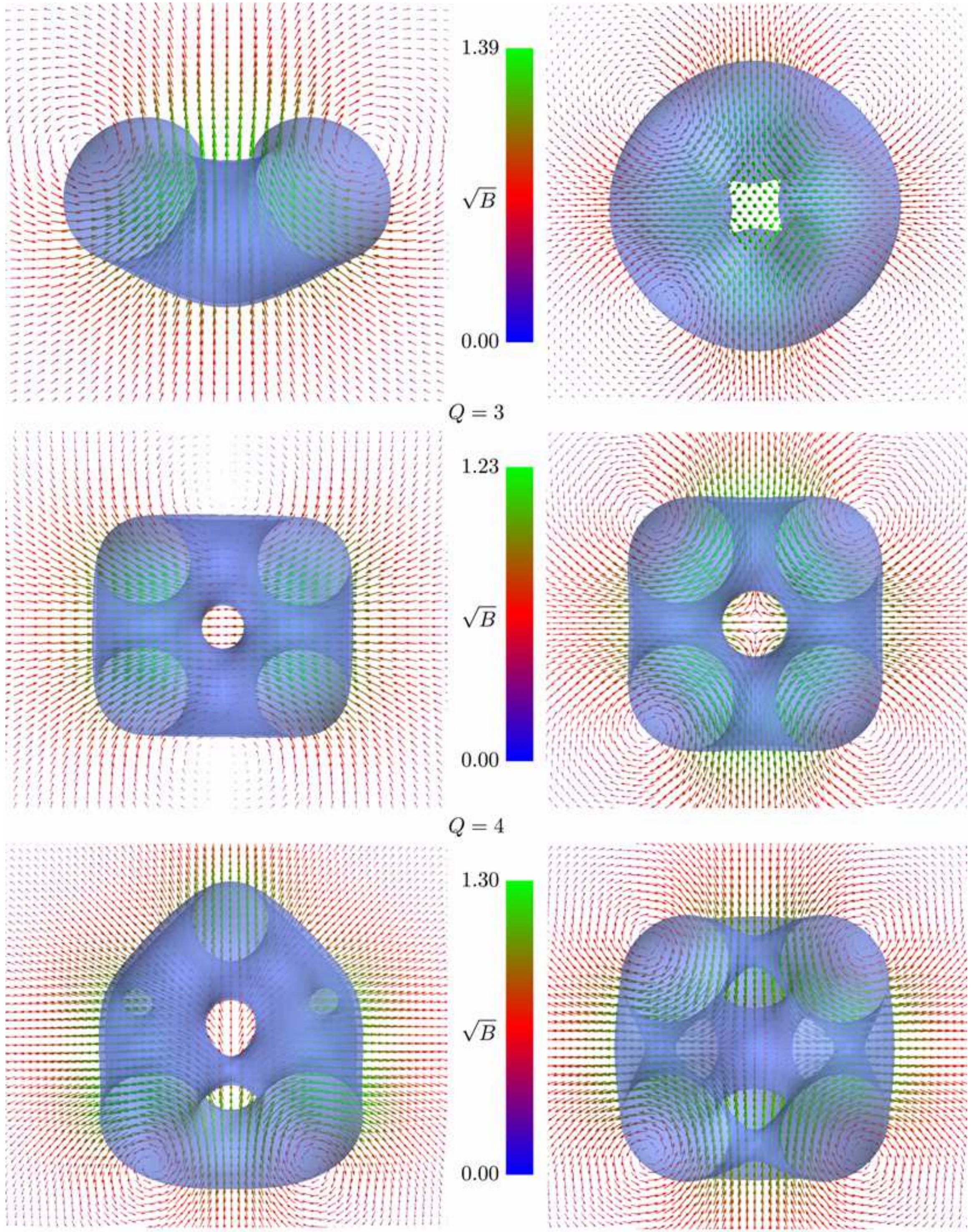


Q=4



Q=5




1.39
$\sqrt{B}$
0.00
$Q = 3$
1.23
$\sqrt{B}$
0.00
$Q = 4$
1.30
$\sqrt{B}$
0.00
$Q = 5$

Gauged Skyrmions = Embedded Hopfions ?

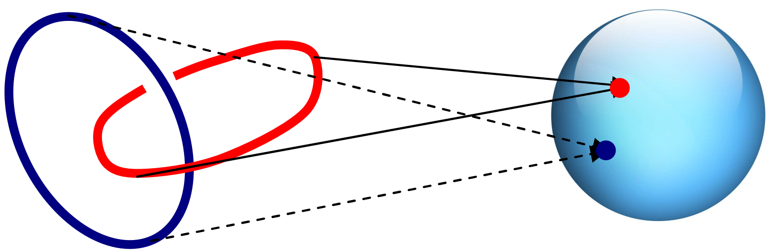
Axially symmetric parametrization:

$$\pi_1 = \psi_1(r,\theta)\cos(n\varphi); \quad \pi_2 = \psi_1(r,\theta)\sin(n\varphi); \quad \pi_3 = \psi_2(r,\theta); \quad \phi_0 = \psi_3(r,\theta)$$

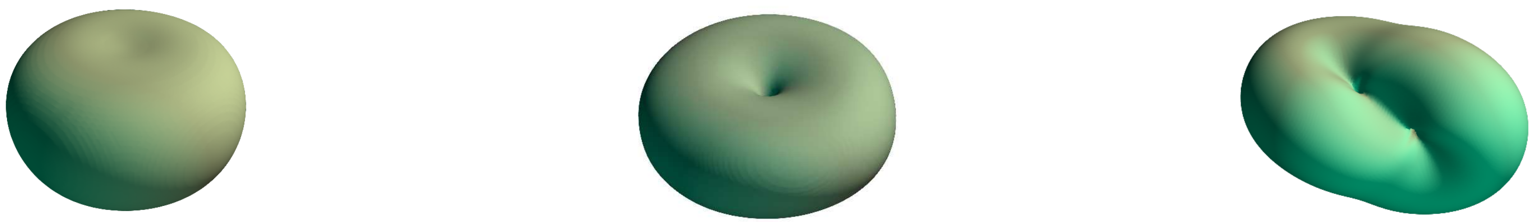
$$\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$$

$\vec{\psi} : S^3 \to S^2$ **1st Hopf map**

Loops in domain space S^3



Target space S^2

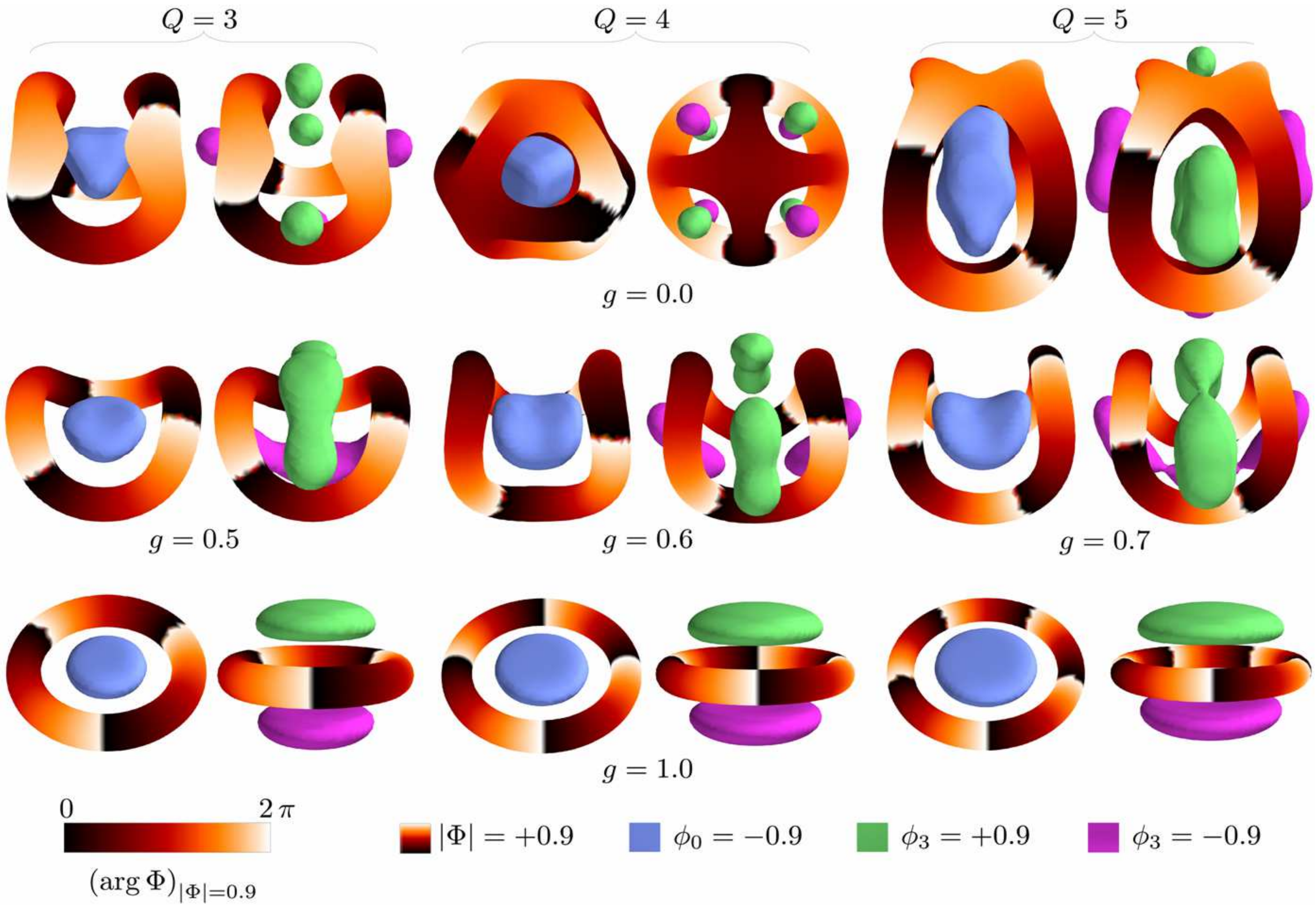


Q=1 $1\mathcal{A}_{1,1}$

Q=2 $2\mathcal{A}_{2,1}$

Q=3 $3\widetilde{\mathcal{A}}_{3,1}$

Gauged Skyrmions = Embedded Hopfions ?



Summary

- There are new multisoliton solutions of the U(1) gauged Skyrme-Maxwell theory.
- The domain of existence is restricted by the condition $|gV| \leq m$.
- There are two critical limits
    - $|g| \gg |V|$ (magnetic limit)
    - $|V| \gg |g|$ (electrostatic limit)
- Strong coupling $\rightarrow$ axial symmetry for all solutions
- Maxwell term alone cannot stabilize the Skyrmions
- Infinitely strong coupling $\rightarrow$ Skyrmion string (Abelian Higgs limit ?)
- Electromagnetic interaction increases the binding energy
- p/n mass splitting and $\pi_{\pm}/\,\pi_0$ mass splitting cannot be expained in the conventional Skyrme-Maxwell model
- U(1) gauged gravitating Skyrmions, hairy black holes, interactions between charged Skyrmions coupled to magnetic fluxes … etc

Thank you!